Unbounded Bivariant K-theory and an Approach to Noncommutative Fréchet Spaces

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vorgelegt von
Nikolay Ivankov
aus
Kaliningrad, Moskau Gebiet (Russland)

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1. Gutachter: Prof. Dr. Yuri I. Manin
2. Gutachter: Prof. Dr. Werner Ballmann

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Abstract

In the current work we thread the problems of smoothness in non-commutative C^*-algebras arising form the Baaj-Julg picture of the KK-theory. We introduce the notion of smoothness based on the pre-C^*-subalgebras of C^*-algebras endowed with the structure of an operator algebra. We prove that the notion of smoothness introduced in the paper may then be used for simplification of calculations in classical KK-theory.

The dissertation consists of two main parts, discussed in chapters 1 and 2 respectively.

In the Chapter 1 we first give a brief overview to Baaj-Julg picture of KK-theory and its relation to the classical KK-theory, as well as an approach to smoothness in Banach algebras, introduced by Cuntz and Quillen. The rest of the chapter is devoted to operator spaces, operator algebras and operator modules. We introduce the notion of stuffed modules, that will be used for the construction of smooth modules, and study their properties. This part also contains an original research, devoted to characterization of operator algebras with a completely bounded anti-isomorphism (an analogue of involution).

In Chapter 2 we introduce the notion of smooth system over a not necessarily commutative C^*-algebra and establish the relation of this definition of smoothness to the Baaj-Julg picture of KK-theory. For that we define the notion of fréchetization as a way of construction of a smooth system form a given unbounded KK-cycle. For a given smooth system $\mathscr{A}$ on a C^*-algebra A we define the set $\Psi_\mu^{(n)}(\mathscr{A}, B)$, $n \in \mathbb{N} \cup \{\infty\}$ of the unbounded (A, B)-KK-cycles that are n smooth with respect to the smooth system $\mathscr{A}$ on A and fréchetization μ. Then we subsequently prove two main results of the dissertation. The first one shows that for a certain class of fréchetizations it holds that for any set of C^*-algebras Λ there exists a smooth system $\mathscr{A}$ on A such that there is a natural surjective map $\Psi_\mu^{(\infty)}(\mathscr{A}, B) \to \mathrm{KK}(A, B)$ for all $B \in \Lambda$. The other main result is a generalization of the theorem by Bram Mesland on the product of unbounded KK-cycles. We also present the prospects for the further development of the theory.

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Contents

Chapter 0

Introduction

The main theme of the present paper, as it follows from the title, is concerned with the smoothness in noncommutative C^*-algebras and the relation of this notion of smoothness to the unbounded bivariant KK-theory.

Historically, the notion of smooth functions on a smooth manifold is given in a more or less canonical way. Namely, there is a standard notion of the algebras $C^n(\mathbb{R}^m)$ of n-differentiable functions on m-dimensional Euclidean space. Then the definition of C^n-smooth manifold is given in terms of smooth functions on $\mathbb{R}^m$: we introduce an atlas on the topological manifold and demand the transition functions between the local charts to be smooth. The algebra $C^k(X)$ of C^k-smooth functions on a C^n-smooth manifold X for $k \leq n$ is then the algebra of all such functions $f \in C(X)$ that are smooth on all the local charts of the chosen C^n-smooth atlas on X. Here we assume that the closures of the open sets constituting the atlas are compact.

The definition of C^∞ smooth manifold and $C^\infty(X)$ is given analogously to the C^n case.

The procedure of defining a smooth manifold structure on a topological manifold is more or less canonical. Of course, it depends on the atlas, but, although there are, for instance, 28 different "exotic" structures of smooth manifold on a 7-dimensional sphere, these are all the structures of the smooth manifold (up to a diffeomorphism) that we may obtain on this particular object. We also recall that the structure of smooth manifold is unique for topological manifolds of dimension ≤ 3.

In turn, the structure of smooth manifold on a topological space X allows us to introduce a tangent space, a Riemann metric and, finally, a spinor bundle and a Dirac operator on X in case when X can be endowed with the spin-manifold structure.

When we switch to the noncommutative geometry, this bottom-up paradigm - from $\mathbb{R}^m$ to smooth manifolds to Riemann manifolds to spin-manifolds - fails to work, because in general there is even no topological space corresponding to a noncommutative C^*-algebra, left alone the local charts on this space. However, many notions arising in differential geometry are generalized for noncommutative geometry using the top-down paradigm.

One of the most well-known examples of such kind of generalization are spectral triples introduced by Alain Connes (the construction outlined, for instance, in [14, IV.4]).

We recall that in the most general case a spectral triple is a set of data $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a dense subalgebra of a C^*-algebra, faithfully represented on a Hilbert space $\mathcal{H}$, and D is a densely defined selfadjoint operator on $\mathcal{H}$, satisfying

- $(1 + D^2)^{-\frac{1}{2}}$ extends to a compact operator on $\mathcal{H}$.
- $[D; a]$ extends to a bounded operator on $\mathcal{H}$

(In the terms of KK-theory, a spectral triple is then an unbounded $(A, \mathbb{C})$-KK-cycle $(\mathcal{H}, D)$). Here the Hilbert space $\mathcal{H}$ plays a role of a "noncommutative spinor bundle" and D act as an analogue of a Dirac operator. One then defines an analogue of "smooth sections of spinor bundle" $\mathcal{H}_\infty = \bigcap_{n=1}^{\infty} \operatorname{Dom} D^n$. The subalgebra $\mathcal{A}$ plays a role of "smooth" functions on A: it is demanded that each element $a \in \mathcal{A}$ restricts to a map $a \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$. Alain Connes introduces the so-called regularity axiom on A: for all $a \in \mathcal{A}$ both a and $[D, a]$ belong to the domain of smoothness $\operatorname{Dom}(\delta^k)$, where $\delta(T) = [|D|; T]$ for $T \in \mathbb{B}(\mathcal{H})$. Additional axioms (such as claiming $\mathcal{H}_\infty$ to be a finitely generated projective $\mathcal{A}$-module) may be imposed to make a spectral triple resemble a differential manifold with a spin-structure. It has been proved by Connes in [17] that every so-called *real spectral triple* (see for instance [34] for definition) corresponds to a spin-manifold whenever the C^*-algebra A is commutative.

Thus, while in differential geometry a spinor bundle and the Dirac operator on it are constructed by means of smooth functions on a smooth manifold, in noncommutative geometry we may go the opposite way: first we choose a "bundle" and a Dirac-type operator, and then this data is used for the definition of smooth sections of the bundle and then smooth subalgebras of a C^*-algebra.

There is an another approach that was proposed by Blackadar and Cuntz in [6]. In this approach the authors tried to simulate the Fréchet spaces with Fréchet seminorms. Again, unlike the differential geometry, where the Fréchet seminorms are defined by means of the supremum norms of partial derivations of smooth functions, the authors applied an abstract Banach space approach. Given a C^*-algebra A, they introduce a so-called *differential seminorm*, which is a system of seminorms with particular condition, and then prove that the dense subalgebras of A, complete with respect to these seminorms, have the properties analogous to the ones of the subalgebras of smooth functions on smooth manifolds. In particular, they are stable under holomorphic functional calculus on A. We shall briefly discuss this approach in the Subsection 1.1.6.

The unbounded KK-theory, first proposed by Saad Baaj and Pierre Julg [2], is similar to noncommutative geometry and has close origins. The main difference is that instead of Hilbert spaces, as in spectral triples, one deals with Hilbert C^*-modules over some C^*-algebra B, and the Dirac-type operators are replaced with so-called unbounded regular operators, which are B-linear (we give the precise definition in Subsection 1.1.5). The unbounded KK-cycles, with spectral triples being their particular case for $B = \mathbb{C}$, were the main object of study of Bram Mesland in his PhD thesis and [28], and apparently are the main object of study of the present paper.

For his studies, Mesland has proposed the approach of smoothness which is similar to the one adopted by Connes. For a given C^*-algebra, he chooses a decreasing nested sequence

$$(\mathcal{A}_\infty \subseteq) \cdots \subseteq \mathcal{A}_n \subseteq \mathcal{A}_{n-1} \subseteq \cdots \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_0 := A$$

of dense subalgebras of A, stable under holomorphic functional calculus on A. This sequence was supposed to be previously given, and was actually claimed to come out of some spectral triple of the form $(\mathcal{A}_\infty, \mathcal{H}, D)$. Then, given an unbounded (A, B)-KK-cycle (E, D), satisfying certain compatibility conditions, Mesland defines a structure of operator algebra on each $\mathcal{A}_n$. This operator algebra structure is then used for Mesland's generalization of Kasparov product in unbounded bivariant K-theory.

In present paper, we introduce yet another one notion of smoothness. In our definition, by a smooth system on a C^*-algebra A we shall understand the sequence $\mathscr{A}$ of operator algebras

$$(\ldots \hookrightarrow)\mathcal{A}_n \hookrightarrow \mathcal{A}_{n-1} \hookrightarrow \ldots \hookrightarrow \mathcal{A}_1 \hookrightarrow \mathcal{A}_0 := A$$

such that all the maps are completely bounded essential inclusions, the images of $\mathcal{A}^{(n)}$ in A are dense and stable under holomorphic functional calculus, and the involution on A induces a completely isometric anti-isomorphism on $\mathcal{A}^{(n)}$. We also introduce a class of operations that we call *fréchetizations*, which, roughly speaking, are the ways μ to define a smooth system $\mathscr{A}_{\mu,D}$ on a given C^*-algebra A by a specified unbounded (A, B)-KK-cycle (E, D). The method of endowing the algebras $\mathcal{A}_n$ with an operator algebra structure proposed in [28] becomes a particular example of fréchetizations, called *mes*-fréchetization.

Then, for given fréchetization μ we define the sets $\Psi_\mu^{(n)}(\mathscr{A}, B)$ of unbounded (A, B)-KK-cycles that are n-smooth (with n possible infinite) relatively to the smooth system $\mathscr{A}$, and prove that for a certain kind of fréchetizations (including *mes*) we may construct the smooth system $\mathscr{A}$ in such a way that for any given n-set of C^*-algebras Λ there is a well defined surjective map $\Psi_\mu^{(n)}(\mathscr{A}, B) \to \mathrm{KK}(A, B)$ for all $B \in \Lambda$.

Alongside with that, show the interesting smooth systems may not necessarily come out from spectral triples, and, from the other hand, that the systems that are coming from spectral triples do not necessarily possess the same properties as systems of Fréchet algebras on Riemann manifolds.

The main purpose of introducing the smooth systems the way we have just described was the generalization of Kasparov product to the unbounded KK-theory. This task was considered by Mesland in [28]. There has been presented a way to construct the product (A, C)-KK-cycle of two unbounded (A, B)- and (B, C)-KK-cycles (E, T) and (Y, D) respectively. However, by the formulation proposed in [28], when dealing with this kind of product, one had always to impose the conditions on the module E and the operator T, that were coming out of the properties of the smooth system induced on the algebra B by the unbounded KK-cycle (Y, D); in the notation we introduce in the current paper this system is denoted by $\mathscr{B}_{mes,D}$. In particular, one has to care about the so-called *smoothness* of the module E with respect to $\mathscr{B}_{mes,D}$ and *transversality* of the operator T. We redefine these conditions in terms of the more general smooth systems introduced above, and prove that if the data (E, T) satisfies these generalized conditions for the system $\mathscr{B}$, then so it does with respect to the smooth systems of the form $\mathscr{B}_{\mu,D}$ for all $(Y, D) \in \Psi^{(\bullet)}(B, C)$.

Thus, we obtain a generalization of the main result of [28], allowing us to calculate the unbounded version of Kasparov product for sets of unbounded KK-cycles rather than just single given pairs of them.

The paper contains several examples illustrating the proposed theory. It also contains an original result threating an analogue of the notion of involution for operator algebras, which could be interesting on its own.

Chapter 1

Preliminaries

1.1 Preliminaries

1.1.1 Hilbert C^*-Modules

Definition 1.1.1. Let A be a C^*-algebra. A complex vector space E with a right A-module structure will be called a *Hilbert C^*-module* if it is equipped with a bilinear pairing

$$\begin{array}{rcl} E \times E & \to & B \\ (\xi, \eta) & \mapsto & \langle \xi, \eta \rangle \end{array}$$

satisfying

- $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$
- $\langle \xi, \eta a \rangle = \langle \xi, \eta \rangle \, a$
- $\langle \xi, \xi \rangle \geq 0$ and $\langle \xi, \xi \rangle = 0 \Leftrightarrow \xi = 0$
- E is complete in the norm $\|\xi\| := \sqrt{\| \langle \xi, \xi \rangle \|}$.

Hilbert C^*-modules serve as natural generalizations of Hilbert spaces, with the pairing on them being an analogue of scalar product. Hilbert spaces may be regarded as Hilbert C^*-modules over $\mathbb{C}$.

The theory of Hilbert C^*-modules is a deep and widely-developed subject of mathematics. We refer to [27] and [26] for detailed exposition of the theory. Here we shall only mention some distinctive features of Hilbert C^*-modules.

A C^*-algebra A is a Hilbert C^*-module over itself with the scalar product given by $\langle a, b \rangle = a^* b$ for $a, b \in A$.

The space $\overline{\operatorname{span}\{\langle \xi, \eta \rangle \mid \xi, \eta \in E\}}$, where the completion is taken with respect to the C^*-norm on A forms an ideal in A. If this ideal coincides with A, then the module is called *full*.

A *direct sum* of two Hilbert C^*-modules E_1 and E_2 is a Hilbert C^*-module, with the scalar product given by

$$\langle \xi_1 \oplus \xi_2, \eta_1 \oplus \eta_2 \rangle = \langle \xi_1, \eta_1 \rangle + \langle \xi_2, \eta_2 \rangle$$

for $\xi_j, \eta_j \in E_j$, $j = 1, 2$.

More generally, for a countable set $\{E_j\}_{j=1}^{\infty}$ of Hilbert C^*-A-modules we may form a direct sum Hilbert C^*-A-module $\bigoplus_{j=1}^{\infty} E_j$ given by the closure of algebraic direct sum of E_j's with respect to the norm, obtained form the pairing

$$\left\langle \oplus_{j=1}^{\infty} \xi_j, \oplus_{j=1}^{\infty} \eta_j \right\rangle := \sum_{j=1}^{\infty} \left\langle \xi_j, \eta_j \right\rangle$$

This pairing also defines the structure of Hilbert C^*-module on $\bigoplus_{j=1}^{\infty} E_j$.

A submodule F of Hilbert C^*-module E is called *orthogonally complementable* if there exists a Hilbert C^*-submodule $F^{\perp} \subseteq E$ such that $F \oplus F^{\perp} = E$. Not all Hilbert C^*-submodules of a given Hilbert C^*-module are orthogonally complementable.

A *standard Hilbert C^*-module* over a C^*-algebra A is obtained as a sum of countable number of copies of algebra A, $\mathcal{H}_A := A \oplus A \oplus A \oplus \ldots$. A distinctive property of Hilbert C^*-modules is given by the so-called Kasparov stabilization theorem.

Theorem 1.1.2. *Let E be a countably generated Hilbert C^*-A-module. Then there is an isometric isomorphism of Hilbert C^*-A-modules.*

$$E \oplus \mathcal{H}_A \approxeq \mathcal{H}_A$$

As a result, every countably generated Hilbert C^*-A-module may be regarded as an orthogonally complementable Hilbert C^*-A-submodule of $\mathcal{H}_A$.

Analogously, one may define the module of the form $\mathcal{H}_E := E \oplus E \oplus \ldots$. There is a distinctive characteristic of full Hilbert C^*-modules:

Theorem 1.1.3 ([26]). *Let E be Hilbert C^*-A-module, which is full. Then $\mathcal{H}_E = \mathcal{H}_A \oplus M$, where M is some Hilbert C^*-A-module. If A is unital, then there exists such $m \in \mathbb{N}$, that $E^m \approxeq A \oplus M$. In case when E is countably generated, we also have that $\mathcal{H}_E \approxeq \mathcal{H}_A$.*

Let E, F be two Hilbert C^*-A-modules. We denote by $\mathrm{Hom}_A(E, F)$ the Banach space of bounded A-linear maps. If $E = F$ we denote $\mathrm{End}_A(E) := \mathrm{Hom}_A(E, E)$.

Unlike the operators on Hilbert space, the A-linear operators on a Hilbert C^*-A-module need not be adjointable. We say, that the operator $T \in \mathrm{Hom}_A(E, F)$ is *adjointable* if there exists such an operator $T^* \in \mathrm{End}_A(F, E)$ such that

$$\langle T\xi, \eta \rangle_F = \langle \xi, T^*\eta \rangle_E$$

for all $\xi \in E$ and $\eta \in F$. The set of all adjointable operators in $\mathrm{End}_A E$ forms an algebra $\mathrm{End}_A^*(E)$, which is a C^*-algebra with the conjugation operation given by $*\colon T \mapsto T^*$.

A finitely generated Hilbert C^*-A-module E is orthogonally complementable in a finitely generated free A-module A^m, $m \in \mathbb{N}$ iff it is projective, i.e. there exists an operator $p \in \mathrm{End}^*_A(E)$ such that $p = p^2 = p^*$, and $E \approxeq pA^m$ as right Hilbert C^*-A-modules.

There is a generalization of compact operators for Hilbert C^*-modules. For given two elements $\xi, \eta \in E$ we define an elementary operator $\xi_\eta(\cdot) = \xi \langle \eta, \cdot \rangle$. The span of all such operators forms an algebra which we denote as $\mathrm{Fin}_B(E) \subseteq \mathrm{End}^*_B(E)$. the completion of $\mathrm{Fin}_B(E)$ with respect to the C^*-norm on $\mathrm{End}^*_B(E)$ gives an algebra denoted as $\mathbb{K}_B(E) \subseteq \mathrm{End}^*_B(E)$, which is called the *algebra of B-compact operators on E*.

A C^*-algebra A is called *$\mathbb{Z}/2\mathbb{Z}$-graded* (we shall call it just *graded*) if there is an element $\hat{\gamma} \in \mathrm{Aut}^*(A)$ of order 2. If the grading is present, then there is a decomposition of $A = A^0 \oplus A^1$, where A^0 is the algebra of *even* elements and the closed subspace A^1 of *odd* elements. It holds that $A^i A^j \subseteq A^{i+j}$, for $i, j \in \mathbb{Z}/2\mathbb{Z}$. A $*$-homomorphism $\phi\colon A \to B$ is called *graded* if it respects grading, that is $\phi \circ \hat{\gamma}_A = \hat{\gamma}_B \circ \phi$. For $a \in A^j$ we denote by $\partial a \in \mathbb{Z}/2\mathbb{Z}$ the degree of a.

Definition 1.1.4. A Hilbert C^*-A-module E is called *graded* if it is equipped with an element $\gamma \in \mathrm{Aut}_{\mathbb{C}}(E)$ of order 2, such that

- $\gamma(\xi a) = \gamma(\xi)\hat{\gamma}(a)$
- $\langle \gamma(\xi), \gamma(\eta) \rangle = \hat{\gamma} \langle \xi, \eta \rangle$

In this case E decomposes in two subspaces $E^0 \oplus E^1$, and $E^i A^j \subseteq E^{i+j}$. The grading on E naturally induces the grading on the algebras $\mathrm{End}_A(E)$, $\mathrm{End}^*_A(E)$ and $\mathbb{K}_A(E)$ by setting $(\hat{\gamma}T)\xi = \gamma(T\gamma(\xi))$.

Throughout the paper we assume both algebras and modules be graded, possibly trivially, i.e. with $\hat{\gamma} = \mathrm{Id}_A$ and $\gamma = \mathrm{Id}_E$.

1.1.2 Tensor products on Hilbert C^*-modules

Let A and B be two graded C^*-algebras. The algebraic tensor product of these two algebras $A\tilde{\otimes}_B := A \otimes_{\mathbb{C}} B$ (we shall always write $\times$ for $\times_{\mathbb{C}}$) may be regarded as a graded algebra subject to the multiplication law given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{\partial b_1 \partial a_2} a_1 a_2 \otimes b_1 b_2$$

According to the Gelfand-Naïmark-Segal construction, there is a graded representation of C^*-algebras A and B on some graded Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Denote by $A\overline{\otimes}B$ the closure of $A \otimes B$ with respect to the norm induced by the representation of $A \otimes B$ in $\mathbb{B}(\mathcal{H} \otimes \mathcal{K})$. We obtain that $A\overline{\otimes}B$ is a graded C^*-algebra with respect to this norm. The C^*-algebra $A\overline{\otimes}B$ is then called *minimal* or *spatial* tensor product of A and B.

Let now E and F be graded C^*-modules over A and B respectively. We may define an $A\overline{\otimes}B$-valued inner product on the algebraic tensor product $E \otimes F$ by setting

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \langle \xi_1, \xi_2 \rangle \otimes \langle \eta_1, \eta_2 \rangle$$

The completion of $E \otimes F$ in the norm induced by this tensor product becomes a Hilbert C^*-$A\overline{\otimes}B$-module, which we denote by $E\overline{\otimes}B$. It also inherits the grading by setting $\gamma_{E\overline{\otimes}F} := \gamma_E \otimes \gamma_F$. The module $E\overline{\otimes}F$ is called *exterior tensor product* of E and F.

If $\phi \in \mathrm{End}^*_A E$ and $\psi \in \mathrm{End}^*_B F$, we may define a *graded tensor product* of these two maps by setting

$$\phi \otimes \psi(\xi \otimes \eta) := (-1)^{\partial\xi\partial\psi}\phi(\xi) \otimes \psi(\eta)$$

This tensor product gives a graded inclusion

$$\mathrm{End}^*_A(E)\overline{\otimes}\mathrm{End}^*_B(F) \to \mathrm{End}^*_{A\overline{\otimes}B}(E\overline{\otimes}F)$$

which may be restricted to an isomorphism

$$\mathbb{K}_A(E)\overline{\otimes}\mathbb{K}_B(F) \to \mathbb{K}_{A\overline{\otimes}B}(E\overline{\otimes}F)$$

Alongside with the exterior tensor product, there is a notion of interior tensor product of Hilbert C^*-modules. It is defined in the following way. We recall that a $*$-homomorphism $A \to \mathrm{End}^*_B(F)$ is called *essential* if the set $AF := \mathrm{span}\{a\eta \mid a \in A, \eta \in F\}$ is dense in F. Given such an essential graded $*$-homomorphism, we may define a pairing on the algebraic tensor product $E \tilde{\otimes}_A F$, where E is a Hilbert C^*-A-module, given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle := \langle \eta_1, \langle \xi_1, \xi_2\rangle\, \eta_2\rangle$$

We denote by $E \tilde{\otimes}_A F$ the completion of $E \tilde{\otimes}_A F$ with the norm induced by this pairing. $E \tilde{\otimes} F$ has a natural structure of Hilbert C^*-B-module.

One may define a $*$-homomorphism

$$\begin{array}{rcl} \mathrm{End}^*_A(E) & \to & \mathrm{End}^*_B(E \tilde{\otimes}_A F) \\ T & \mapsto & T \otimes 1 \end{array}$$

This $*$-homomorphism restricts to $\mathbb{K}_A(E) \to \mathbb{K}_B(E \tilde{\otimes}_A F)$. The module $E \tilde{\otimes}_A F$ will also carry an essential representation of A. We shall also denote this product by $E \tilde{\otimes}_\pi F$ to specify the representation $\pi\colon\ A \to \mathrm{End}^*_B(F)$.

1.1.3 Regular Unbounded Operators on Hilbert C^*-modules

We follow [28] in exposition of unbounded regular operators

Definition 1.1.5 ([2])**.** Let E be a C^*-A-module. A densely defined closed operator $D\colon \mathrm{Dom}D \to E$ is called *regular* if

- D^* is densely defined on E
- $1 + D^*D$ has a dense range

It follows from the definition that regular operators are B-linear and $\mathrm{Dom} D$ is a B-submodule of E. There are two operations, that are canonically associated with an unbounded operator D. The first one is the *resolvent*,

$$\mathfrak{r}(D) := (1 + D^*D)^{-\frac{1}{2}}$$

The other one is called *bounded transform*, also known as *z-transform* of D.

$$\mathfrak{b}(D) := D(1 + D^*D)^{-\frac{1}{2}}$$

Both operators are densely defined on E and extend to the elements of $\mathrm{End}^*_A(E)$.

A regular operator is called *symmetric* if $\mathrm{Dom} D \subseteq \mathrm{Dom} D^*$ and $D = D^*$ on $\mathrm{Dom} D$. It is *selfadjoint* if it is symmetric and $\mathrm{Dom} D = \mathrm{Dom} D^*$

Proposition 1.1.6. *If $D\colon \mathrm{Dom} D \to E$ is regular, then D^*D is selfadjoint and regular. Moreover, $\mathrm{Dom} D^*D$ is a core of D and $\mathrm{Im}\,\mathfrak{r}(D) = \mathrm{Dom} D$.*

The bounded transform operation may be reversed in a sense that the unbounded regular operator D may be fully recovered from its bounded transform $\mathfrak{b}(D)$ by the formula

$$D = \mathfrak{b}(D)(1 - \mathfrak{b}(D)^*\mathfrak{b}(D))^{-\frac{1}{2}}$$

By the *graph* of E we shall understand the closed submodule

$$\mathfrak{G}(D) := \{(\xi, D\xi) \mid \xi \in \mathrm{Dom}(D)\} \subseteq E \oplus E$$

There is a canonical unitary $v \in \mathrm{End}^*_A(E \oplus E)$, given by $v(\xi, \eta) := (-\eta, \xi)$. We note that the modules $\mathfrak{G}(D)$ and $v\mathfrak{G}(D^*)$ are orthogonal submodules of $E \oplus E$. Woronowicz presents in [35] an algebraic characterization of regularity for unbounded operators:

Theorem 1.1.7 ([35])**.** *A densely defined closed operator $D\colon E \supseteq \mathrm{Dom} D \to E$ with densely defined adjoint is regular if and only if $\mathfrak{G}(D) \oplus v\mathfrak{G}(D^*) \approxeq E \oplus E$*

This isomorphism is given by a coordinatewise addition. Moreover, the operator

$$p_D := \begin{pmatrix} \mathfrak{r}^2(D) & \mathfrak{r}(D)\mathfrak{b}(D)^* \\ \mathfrak{b}(D)\mathfrak{r}(D) & \mathfrak{b}(D)\mathfrak{b}(D)^* \end{pmatrix}$$

is a projection on $E \oplus E$, and $p_D(E \oplus E) = \mathfrak{G}(D)$. We shall refer to this projection operator as *Woronowicz projection*.

It should also be noted that if D is an odd operator on E, then the grading $\gamma \oplus (-\gamma)$ on $E \oplus E$ respects the decomposition form the Theorem 1.1.7.

Since there is a bijection between $\mathfrak{G}(D)$ and $\mathrm{Dom}(D)$, and the latter is a submodule of $E \oplus E$, we may naturally equip $\mathfrak{G}(D)$ with the structure of Hilbert C^*-A-module.

We observe that when D is felfadjoint, it commutes with both $\mathfrak{r}(D)$. Abusively denoting by

$$D := \mathrm{diag}_2(D)\colon \mathrm{Dom} D \oplus \mathrm{Dom} D \to E \oplus E$$

and

$$\mathfrak{r}(D) := \mathfrak{r}(\mathrm{diag}_2(D)) = \mathrm{diag}_2(\mathfrak{r}(D))\colon E \in \mathrm{End}^*_B(E \oplus E)$$

we obtain that D maps $\mathfrak{r}(D)\mathfrak{G}(D)$ to $\mathfrak{G}(D)$. Denote the restriction of D to $\mathfrak{G}(D)$ by D_2. We have the following result:

Theorem 1.1.8 ([28]). *Let $D\colon \mathrm{Dom}D \to E$ be a selfadjoint regular operator. Then $D_2\colon \mathfrak{r}(D)\mathfrak{G}(D) \to \mathfrak{G}(D)$ is also a selgfadjoint regular operator. When D is odd, so is D_2*

The operation may be proceeded to obtain D_3 from D_2 the same way as we have obtained D_2 form D, and so on. As a result, we have the following

Corollary 1.1.9 ([28]). *A selfadjoint regular operator $D\colon \mathrm{Dom}D \to E$ induces a morphism of inverse systems of C^*-modules:*

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & E_{n+1} & \longrightarrow & E_n & \longrightarrow & E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & E \\
 & \searrow^{D_{n+1}} & & \searrow^{D_n} & & \searrow^{D_{n-1}} & & \searrow^{D_{n-2}} & & \searrow^{D_2} & & \searrow^{D_1=D} & \\
\cdots & \longrightarrow & E_{n+1} & \longrightarrow & E_n & \longrightarrow & E_{n-1} & \longrightarrow & \cdots & \longrightarrow & E_1 & \longrightarrow & E
\end{array}$$

Here $E_n = \mathfrak{G}(D_n)$, and the maps represented by horizontal and diagonal arrows are projections of $\mathfrak{G}(D_n)$ on the first and the second copy of $\mathfrak{G}(D_{n-1})$ respectively.

Following [28], we shall refer to this chain as *Sobolev chain* of D.

1.1.4 KK-Theory

The KK-theory, also known as *bivariant* K-*theory* or *Kasparov* K-*theory* was developed by Gennady Kasparov in early 80'th as a generalization of both K-theory and K-homology and was supposed to be a tool for finding the answer, whether the so-called Novikov's Conjecture holds. Thereafter the theory proved itself to be an important tool for different theoretical means, including theoretical physics (D-brane theory).

We shall briefly outline the construction of Kasparov KK-groups.

Definition 1.1.10. Let, A and B be C^*-algebras. An (A, B)-KK-cycle is a triple (E, π, F) where

- E is a countably generated graded Hilbert C^*-B-module.
- $\pi\colon A \to \mathrm{End}^*_B(E)$ is a graded representation of A on E
- F is a Fredholm operator on E, such that $(F^2 - 1)\pi(a) \in \mathbb{K}_B(E)$, $[F; \pi(a)]$ and $(F - F^*)\pi(a)$ all lay in $\mathbb{K}_B(E)$ for each $a \in A$.

The set of all (A, B)-KK-cycles is denoted by $\mathbb{E}_0(A, B)$.

The set $\mathbb{E}_0(A, B)$ has a natural semigroup structure given by the direct sum

$$(E_1, \pi_1, F_1) \oplus (E_2, \pi_2, F_2) := (E_1 \oplus E_2, \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}, \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix})$$

Now we give an equivalence relations on KK-cycles, that will allow us to define the KK-groups.

Definition 1.1.11. Two (A,B)-KK-cycles (E_i,π_i,F_i), $i=1,2$ are called *unitary equivalent* if there is a (grading preserving) unitary in $U\in\mathbb{B}(E_1,E_2)$ such that

$$(E_2,\pi_2,F_2)=(E_2,U\pi_1U^{-1},UF_1U^{-1})$$

We denote this equivalence relation by $\sim_u$.

Definition 1.1.12. Two unbounded (A,B)-KK-cycles (E_1,π_1,F_1) and (E_2,π_2,F_2) are called *homotopy equivalent* if there is an $(A,C([0,1])\times B)$-KK-cycle (E,π,F) such that

$$(E\mathbin{\tilde{\otimes}_{f_i}}B, f_i\circ\pi, F\mathbin{\tilde{\otimes}_{f_i}}B)\sim_u(F_i,\pi_i,F_i)$$

where $f_i\colon C([0,1])\times B$ $i=1,2$ are evaluation homomorphisms. We denote this equivalence relation by $\sim_h$.

Finally, we are ready to give the definition of the KK-group.

Definition 1.1.13. For two C^*-algebras A and B we set

$$\mathrm{KK}(A,B):=\mathbb{E}(A,B)/\sim_h$$

One then defines higher KK-groups by setting $\mathrm{KK}_j(A,B):=\mathrm{KK}(A,B\overline{\otimes}\mathbb{C}_j)$, where $\mathbb{C}_j$ is the j-th Clifford algebra. Fortunately, the variant of Bott periodicity theorem holds for KK-groups, so in fact $\mathrm{KK}_j(A,B)=\mathrm{KK}_{j+2k}(A,B)$ for $k\in\mathbb{Z}$, and we deal only with two groups: $\mathrm{KK}_0(A,B)$ and $\mathrm{KK}_1(A,B)$

Remark 1.1.14. If the algebras A and B are assumed to be trivially graded, we may give a more obvious definition of $\mathrm{KK}_1(A,B)$. Namely, one may define the set $(E)_1=\{(E,\pi,F)\}$, such that E, π and F satisfy the conditions of the definition 1.1.10, but without any assumptions on grading and degree of operator F. In this case $\mathrm{KK}_1=\mathbb{E}_1/\sim_h$.

The semigroup operatoin on $\mathbb{E}$ induces a binary operatoin on $\mathrm{KK}(A,B)$, and it may be directly shown that $\mathrm{KK}_i(A,B)$ for $i=0,1$ are actually an Abelian group with respect to this operation.

Remark 1.1.15. We should denote the element of the $\mathrm{KK}(A,B)$ given by the $\sim_h$-equivalence class of a KK-cycle $(E,\pi,F)\in\mathrm{E}(A,B)$ by $[(E,\pi,F)]$.

The KK theory incorporates both K-theory and K-homology. Namely, we have that $\mathrm{KK}_i(\mathbb{C},A)=\mathrm{K}_i(A)$ and $\mathrm{KK}_i(A,\mathbb{C})=\mathrm{K}^i(A)$ for $i=0,1$.

The KK-groups have many important properties such as additivity and functoriality. However, one of the most important results achieved in the KK-theory is the generalization of the index theorem, also called as Kasparov product. In its most general form the theorem reads as

Theorem 1.1.16. *Let A_1, B_1, A_2, B_2 and C be C^*-algebras. Then there is a well defined product pairing*

$$\mathrm{KK}_i(A_1,B_1\overline{\otimes}C)\otimes_{\mathbb{Z}}\mathrm{KK}_j(A_2\overline{\otimes}C,B_2)\xrightarrow{\tilde{\otimes}_C}\mathrm{KK}_{i+j}(A_1\overline{\otimes}B_1,A_2\overline{\otimes}B_2)$$

One of the most interesting specifications of this result is obtained when we set $B_2 = A_1 = \mathbb{C}$. Then the theorem may be formulated as follows:

Theorem 1.1.17. *[[24]] Let A, B and C be C^*-algebras. Then there is a well defined inner product pairing*

$$\mathrm{KK}_i(A,B) \otimes_{\mathbb{Z}} \mathrm{KK}_j(B,C) \xrightarrow{\otimes_B} \mathrm{KK}_{i+j}(A,C)$$

This pairing is called *Kasparov product* or *internal product in* KK*-theory*. It is associative and, in case when $A = C = \mathbb{C}$ and $B = C(X)$ for some topological manifold X, it coincides with the Atiyah's index map $\mathrm{K}^0(X) \otimes_{\mathbb{Z}} \mathrm{K}_0(X) \to \mathbb{Z} = \mathrm{KK}_0(\mathbb{C},\mathbb{C})$.

Another specification of the Theorem 1.1.16 is obtained when we set $C = \mathbb{C}$. Namely, we have the product of the form

$$\mathrm{KK}_i(A_1,B_1) \otimes_{\mathbb{Z}} \mathrm{KK}_j(A_2,B_2) \xrightarrow{\overline{\otimes}} \mathrm{KK}_{i+j}(A_1 \overline{\otimes} A_2, B_1 \overline{\otimes} B_2)$$

which is also known as *external product in* KK*-theory*.

But, although the existence of the product form the Theorem 1.1.16 has been proved, the calculation of concrete values in this pairing remains a nontrivial task. The main problem is that for given two unbounded KK-cycles $(E_1, \pi_1, F_1) \in \mathbb{E}(A_1, B_1 \overline{\otimes} C)$ and $(E_2, \pi_2, F_2) \in \mathbb{E}(A_2 \overline{\otimes} C, B_2)$ the finding of an element $(E, \pi, F) \in \mathbb{E}(A_1 \overline{\otimes} A_2, B_1 \overline{\otimes} B_2)$ such that

$$[(E_1, \pi_1, F_1)] \otimes_{\mathbb{Z}} [(E_2, \pi_2, F_2)] = [(E, \pi, F)]$$

involves an application of the result known as Kasparov technical lemma [24]. The most complicated computations are concerned with the calculation of the Fredholm operator F.

There have been proposed several methods to avoid using the Kasparov technical lemma. One of them is described in the next subsection and plays a central role in a paper as a whole.

Remark 1.1.18. In literature the notation for the representation π is often suppressed, and the KK-cycles are denoted just as (E, F) instead of (E, π, F). In the following text we shall mostly use this shortened notation.

1.1.5 Unbounded Picture of KK-theory

The unbounded approach to KK-theory was proposed by Saad Baaj and Pierre Julg in [2], published just two years after the Kasparov's original result. In this paper there have been proposed an approach to a KK-theory that sufficiently simplified the calculation of the Kasparov exterior product. The main idea of Baaj and Julg was to replace the Fredholm operator in the definition of the (A, B)-KK-cycle by an unbounded operator. More precisely, we have the following definition:

Definition 1.1.19. Let A and B be C^*-algebras. An *unbounded* (A, B)-KK-*cycle* is a triple (E, π, D) with E and π as in the Definition 1.1.10, and D is a selfadjoint regular unbounded operator on E, satisfying

- $\mathfrak{r}(D)\pi(a) \in \mathbb{K}_B(E)$ for all $a \in A$;
- The set of all such $a \in A$, that $[D; \pi(a)]$ extends to a bounded operator on E, is dense in A.

Remark 1.1.20. As in the previous subsection, we shall suppress the notation for the representation π and write just (E, D) for an unbounded KK-cycle.

Historically, an approach of Baaj and Julg was a step back to the origins of the KK-theory. The main point of their suggestion was that the conditions on an unbounded operator in the Definition 1.1.19 are in fact the conditions which hold for a pseudodifferential operator on a Riemann manifold. However, when calculating the index of the operator, one have encountered difficulties caused by the fact that the degree of the differential operator should always be taken into consideration. It was due to Atiyah, who has proposed in [1] to replace the classical elliptic pseudodifferential operators with the operators of degree 0 (i.e. just bounded), using the operation that was then generalized to yield the bounded transform operation. The resulting operator had the properties described in the Definition 1.1.10. At that stage, the approach of Atiyah simplified the theory employed for the definition of the index map.

The motivating result for the introduction of Baaj-Julg picture was the simplification of the calculations in exterior product in KK-theory. In [2] there has been proved the following

Theorem 1.1.21 ([2]). *Let (E_i, D_i) be unbounded (A_i, B_i)-bimodules for $i = 1, 2$. Then the operator*

$$D_1 \otimes 1 + 1 \otimes D_2 \colon \mathrm{Dom} D_1 \otimes \mathrm{Dom} D_2 \to E_1 \overline{\otimes} E_2 \tag{1.1}$$

extends to a selfadjoint regular operator on $E \otimes F$. Moreover, we have that

$$[(E_1, \mathfrak{b}(D_1))] \times [(E_2, \mathfrak{b}(D_2))] = [(E_2 \overline{\otimes} E_2, D_1 \otimes 1 + 1 \otimes D_2)]$$

as elements of $\mathrm{KK}_j(A_1, B_1)$, $\mathrm{KK}_k(A_1, B_1)$ *and* $\mathrm{KK}_{j+k}(A_1 \overline{\otimes} A_2, B_1 \overline{\otimes} B_2)$ *respectively.*

The simplification here is achieved because now we should just care for the operator $D_1 \otimes 1 + 1 \otimes D_2$ to be selfadjoint and regular; in the bounded picture it is not in general true that $F_1 \otimes 1 + 1 \otimes F_2$ satisfies the requirements of 1.1.10.

In fact, in case when $B_1 = B_2 = \mathbb{C}$, A_1, A_2 are the algebras of continuous functions on a smooth manifolds M and N, and D_1, D_2 are pseudodifferential operators on (some bundles on) M and N respectively, the equation 1.1 coincides with the formula for calculating the "product" of two pseudodifferential operators on the cartesian product of M and N.

One could have been expected, that the Baaj-Julg picture may provide analogous "geometrical" simplifications for the calculation of interior product in KK-theory. However, there have occurred the problems concerned with the fact that in Baaj-Julg picture one may only have a dense subset of $\mathcal{A} \subseteq A$ for which $[D; a]$ extends to a bounded operator on E. This dense subspace plays a role on the algebra of C^1-smooth functions in the C^*-algebra A, though in fact this algebra may have only a very distant relation to C^1-smooth algebras on topological manifolds.

The additional conditions that have to be imposed in order to get through the arising complications are one the main topics of the present paper, and will be discussed in the next chapter. For now, however, we need to give some more preliminaries form Banach and operator algebra theory, that will be then used in the construction.

1.1.6 Holomorphic Stability and Smoothness in Banach Algebras

We recall (cf. [6],[28]), that if $\mathcal{A}$ is an algebra with the Banach norm $\|\cdot\|_\alpha$, and A_α be its closure with respect to this norm, then a Banach norm $\|\cdot\|_\beta$ on $\mathcal{A}$ is called *analytic* with respect to $\|\cdot\|_\alpha$, if for all $a \in \mathcal{A}$ such that $\|a\|_\alpha < 1$ holds

$$\limsup_{n\to\infty} \frac{\ln \|a^n\|_\beta}{n} \leq 0$$

The main consequence of analyticity of one norm with respect to another is the stability of A_β with respect to the holomorphic functional calculus on A_α (cf. [6],[28]). Here A_β is the completion of $\mathcal{A}$ with respect to $\|\cdot\|_\beta$.

Observe also, that if $\|\cdot\|_\gamma \leq C\|\cdot\|_\beta$ then it is also analytic with respect to $\|\cdot\|_\alpha$. Indeed,

$$\limsup_{n\to\infty} \frac{\ln \|a^n\|_\gamma}{n} \leq \limsup_{n\to\infty} \frac{\ln C\|a^n\|_\beta}{n} = \limsup_{n\to\infty} \frac{\ln C + \ln \|a^n\|_\beta}{n} \leq 0 \tag{1.2}$$

The notion of the relative analyticity of the norms may be applied to arbitrary algebras. However, the holomorphic stability of Banach algebras may be obtained in more subtle ways. One of them comes out from the notion of differential seminorm on a C^*-algebras.

Definition 1.1.22 ([6, 4])**.** Let A be a C^*-algebra and X be a dense $*$-subalgebra of A. Denote by ω^+ the set of all nonnegative scalar sequences. A *differential seminorm* on X is a mapping $T\colon X \to \omega^+$, $a \mapsto T(a) = (T_0(a), T_1(a), T_2(a), \dots)$ satisfying the conditions:

1. $T_0(a) \leq c\|a\|$ for all $a \in X$,
2. $T(a+b) \leq T(a) + T(b)$, $T(\lambda a) = |\lambda| T(a)$ for all $x, y \in X$ and all $\lambda \in \mathbb{C}$,
3. $T(ab) \leq T(a)T(b)$ (convolution product),
4. $T(a^*) = T(a)$.

As we have mentioned in the Introduction, the differential seminorms resemble the Fréchet seminorms, generalizing them for noncommutative setting. One of the properties manifesting this resemblance is holomorphic stability of smooth subalgebras.

Let $p_n(a) = \sum_{k=0}^n T_k(a)$ for $n = 0, 1, 2, \dots$. Each p_k is a submultiplicative $*$-seminorm. Define $\mathcal{A}_k$ to be the completion of X in A with respect to p_k; each $\mathcal{A}_k$ is a $*$-Banach algebra. Finally, let $\mathcal{A}_\infty := \operatorname{proj\,lim} \mathcal{A}_n$, which is a Fréchet $*$-algebra.

Theorem 1.1.23 ([3, 4])**.** *$\mathcal{A}_n$ and $\mathcal{A}_\infty$ are C^*-spectral and spectrally invariant in A via the inclusion map.*

Remark 1.1.24. Observe that Theorem 1.1.23 remains true if we replace a differential seminorm T with system of seminorms T', such that T'_n is equivalent to T_n for all $n \in \mathbb{N}$.

Remark 1.1.25. It should be noted, that even if the algebra A is commutative, the differential seminorm may differ sufficiently from the Fréchet seminorms. As a simplest example, consider a 2-torus $\mathbb{T}^2 := \mathbb{R}/(\mathbb{Z} \oplus \mathbb{Z})$, with the differential seminorm

$$T(f(x,y)) := \left(\sup |f(x,y)|, \sup \left|\frac{\partial f(x,y)}{\partial x}\right|, \sup \left|\frac{\partial^2 f(x,y)}{(\partial x)^2}\right|, \dots\right)$$

The algebras $\mathcal{A}_n$ will then consist of the functions that are C^n in the x-direction, but should not be more than continuous in the y-direction.

1.2 Operator Spaces

1.2.1 Concrete Operator Spaces, Completely Bounded Maps

Definition 1.2.1 ([9])**.** A (concrete) *operator space* is a linear subspace Y of $\mathbb{B}(H)$ some Hilbert space H. A (concrete) *operator algebra* is a subalgebra B of $\mathbb{B}(H)$. A (concrete right) *operator A-module* is a subspace Y of $\mathbb{B}(H)$ which is right invariant under the multiplication by elements of A as a subalgebra of $\mathbb{B}(H)$.

Definition 1.2.2 ([9])**.** A linear map $T\colon Y \to Z$ between two operator spaces is *completely bounded* if the map $T \otimes \mathrm{Id}_{\mathbb{K}}\colon Y \overline{\otimes} \mathbb{K} \to Z \overline{\otimes} \mathbb{K}$ is bounded with respect to the spatial norm. For this to hold it is sufficient that the maps $T_n = T \otimes \mathrm{Id}_{M_n}\colon M_n(Y) \to M_n(Z)$ be uniformly bounded, and in this case the smallest bound which works for all n is the norm of $T \otimes \mathrm{Id}_{\mathbb{K}}$. We shall denote this norm by $\|T\|_{cb}$. The set of all *cb*-maps from an operator space E to an operator space E' will be denoted by $\mathrm{CB}(E, E')$. We say that T is *completely contractive* if $\|T\|_{cb} \leq 1$, *completely isometric* or *a complete isometry*, if $T_n = T \otimes \mathrm{Id}_{M_n}$ is an isometry for all n, and *completely bicontinuous* or *a cb-isomorphism* if it is an algebraic isomorphism with T and T^{-1} being completely bounded.

In the algebra (resp. module) case we may of course require the morphisms to be homomorphisms (resp. module maps). One may also define the $\mathbb{Z}/2\mathbb{Z}$grading on operator algebras and operator modules in a standadrd way that we used in Subsection 1.1.1.

In this article we shall also suppose all the operator spaces to be complete as Banach spaces, although it should not necessary hold in general.

The definition of an essential action of an operator algebra on an operator module is analogous to the one in C^*-algebra setting.

Definition 1.2.3. Let A be an operator algebra and E be a (right) operator A-module. The module action of A is called *essential* if EA is dense in E. Otherwise, the *essential subspace* of E for the action of A is the closure of EA.

Remark 1.2.4. In case when the algebra A has a bounded approximate unit, the essentiality of the map is equivalent to the condition that for all $\xi \in E$ there are an element $\xi' \in E$ and $a \in A$ that $\xi = \xi' a$ ([9]). This also means that in fact EA coincides with E.

1.2.2 Abstract Characterizations of Operator Spaces

Alongside with the concrete operator spaces, one may consider the "abstract" ones, which, as we shall see in a moment, characterize concrete operator spaces up to a complete isometry.

Definition 1.2.5. [33, 12] An *$\mathcal{L}^\infty$-matricially normed space* is a pair $\{Y, \{\,_n\|\cdot\|\}\}$, where Y is a vector space over the complex numbers $\mathbb{C}$ and $_n\|\cdot\|$ are norms on $M_n(Y)$, $n \in \mathbb{N}$, satisfying the conditions:

1. $_{n+k}\|x \oplus y\| = \max\{\,_n\|x\|;\,_k\|y\|\}$
2. $\|\alpha x \beta\| \leq \|\alpha\|\,_n\|x\|\|\beta\|$

for all $x \in M_n(Y)$, $x \in M_k(Y)$ and $\alpha, \beta \in M_n(\mathbb{C})$

Throughout the paper we assume that all $\mathcal{L}^\infty$-matricially normed spaces are norm complete.

The result of Ruan and Effros give us the characterization of such spaces.

Theorem 1.2.6 ([21]). *Every $\mathcal{L}^\infty$-matricially normed space is completely isometrically isomorphic to a (concrete) operator space.*

This result also allows us to establish the fact that the space $\mathrm{CB}(X, Y)$, endowed with the *cb*-norm, is completely isometrially isomorphic to an operator space. This is done via the identification $M_n(\mathrm{CB}(X, Y)) \approxeq \mathrm{CB}(X, M_n(Y))$, which assigns matrix norms to $\mathrm{CB}(X, Y)$ (see, for ex. [20]).

1.2.3 Characterizations of Operator (Pseudo)Algebras

For the algebras the situation is less clear then for the modules. In general one may establish the characterization of abstract operator algebras only up to a completely bounded isomorphism. First of all, we need to give the definition of what we shall understand by an abstract operator algebra.

Definition 1.2.7. An *operator pseudoalgebra* is an algebra and assume that A is also an $\mathcal{L}^\infty$ matricially normed space, such that the multiplication map $\mu\colon A \times A \to A$ on A is a completely bounded bilinear map.

This notation is not conventional. Blecher and Le Merdi in [10] use the term *operator K-algebras* where K is the *cb*-norm of the multiplication map. In [33] the term operator pseudoalgebras was used for spaces with completely contractive multiplication map.

There are several results on operator pseudoalgebras that play the same role for the theory of operator algebras as the Gelfand-Na imark-Segal construction plays for the theory of C^*-algebras, that is, establish an isomorphism between pseudoalgebras and concrete operator algebras. The most general result, which we shall use most in the current paper, is due to Blecher:

Theorem 1.2.8 ([7, 10]). *Let A be an algebra and assume that A is also an $\mathcal{L}^\infty$ matricially normed space. Then A may be represented completely boundedly isomorphically as an operator algebra if and only if it is a pseudoalgebra, i.e. its multiplication map is completely bounded.*

As well as for operator algebras, there is a notion of grading for operator pseudoalgebras.

There is a useful observation, that according to [7] the *cb*-isomorphism ρ between the algebra A and an operator algebra constructed in Theorem 1.2.8 could be defined in such a way that $\|\rho\|_{cb} \leq 2K$ and $\|\rho^{-1}\|_{cb} \leq K^{-1}$, where K is a *cb*-norm of the multiplication map in $\mathcal{A}$.

The theorem 1.2.8 will in fact suffice for our needs. However, there are also at least two results indicating the cases when an operator pseudoalgebra is actually completely isometrically isomophic to a concrete operator algebra.

Theorem 1.2.9 ([33]). *Let A be an algebra which is also $\mathcal{L}^\infty$-matricially normed space, and the multiplication on A is completely contractive. Suppose also that there exists a net of elements $\{e_\alpha\}$ in A such that $\mu(a - e_\alpha a) \to 0$ and $\mu(a - a e_\alpha) \to 0$ for all $a \in A$ (contractive approximate identity).Then A is completely isometrically isomorphic to a (concrete) operator algebra with contractive approximate identity.*

The latter theorem is a generalization of the result of Blecher-Ruan-Sinclar, which, in turn, becomes its obvious corollary.

Theorem 1.2.10 ([12]). *In the conditions of previous theorem, suppose that A is unital, that is, there exists an element $e \in A$ such that $\mu(a, e) = \mu(e, a) = a$. Then A is completely isometrically isomorphic to a (concrete) unital operator algebra.*

There is an important fact that in absence of a contractive approximate unit the Theorem 1.2.9 does not work, and we have to retreat to Theorem 1.2.8. Thus, there is not so much difference between operator 1-algebras and operator K-algebras (in notation of [10]) for an arbitrary positive K. Therefore it seems justified to use the notation operator pseudoalgebras for operator K-algebras in general.

1.2.4 Operator Algebras and Involution

The goal of this subsection is to establish the connection between the result 1.2.8 and the involution. We have to point out, that the involutivity is a characteristic of a Banach algebra rather then an operator algebra. However, it plays an important role in many mathematical construction, from which the most relevant to us is smooth Banach algebras introduced by Balckadar and Cuntz ([6], see also [3]).

Recall that an involution on a Banach algebra A is an isometric anti-isomorphism $*\colon A \to A$, $*\colon a \mapsto a^*$ such that $a^{**} = a$.

Thus, if we want to specialize this notion for the case of operator algebras, we should first give a definition of a *cb*-anti-isomorphism.

Definition 1.2.11. Let A be an operator pseudoalgebra. Then an anti-homomorphism $f\colon A \to B$ will be called *cb-anti-homomorphism* if there exists a positive number C such that

$$ {}_n\|(f(a_{ji}))_{ij}| \leq C \ {}_n\|(a_{ij})_{ij}\| $$

for all $(a_{ij})_{ij} \in M_n(A)$. If f is anti-isomorphic, and its inverse f^{-1} is also a *cb*-anti-homomorphism, then f will be called a *cb-anti-isomorphism*. Analogously one may define a completely isometric anti-isomorphism.

Remark 1.2.12. Observe that, unlike the case of homomorphism, we have to add a transposition in matrix algebras to the definition of *cb*-anti-homomorphisms. This makes the notion of *cb*-anti-homomorphism much more subtle then the one of *cb*-homomorphism. It seems, although the author doesn't have a concrete example for now, that even for a general (concrete) operator algebra A there would not be any *cb*-anti-isomorphisms of A onto itself. However, algebras having *cb*-anti-isomorphisms often appear in applications. For instance, the involution on C^*-algebras satisfies this property. The notion of *cb*-anti-isomorphism is also widely used in [28].

Definition 1.2.13. A *cb*-anti-isomorphism $f\colon A \to A$ such that $f^2 = \mathrm{Id}_A$ would be called an *(operator algebra) pseudo-involution* on A. If, in addition, f is completely isometric then it will be called *(operator algebra) involution*. An operator algebra possessing an involution will be called *involutive*.

We are going to show that any pseudo-involution may in some sense be "updated" to become an involution.

Proposition 1.2.14. *Let A be an operator K-algebra with a pseudo-involution f. Then there is an operator pseudoalgebra B and a cb-isomorphism $\sigma\colon A \to B$, such that $\sigma f \sigma^{-1}$ is an involution on B.*

Proof. Let $B = A$ as a algebras We define matrix norms on B as

$$ {}_n\|(a_{ij})_{ij}\|_B = \max\{ {}_n\|(a_{ij})_{ij}\|_A, \ {}_n\|(f(a_{ji})_{ij})\|_A\} $$

The space B endowed with this system of norms is an operator pseudoalgebra. Indeed, we have that

$$ \begin{aligned} &{}_{n+m}\|(a_{ij} \oplus b_{kl})\|_B = \\ &= \max\{\max\{ {}_n\|(a_{ij})_{ij}\|_A, \ {}_n\|(f(a_{ji}))_{ij}\|_A\}, \max\{ {}_m\|(b_{kl})_{lk}\|_A, \ {}_m\|(f(b_{kl}))_{lk}\|_A\}\} \\ &= \max\{\max\{ {}_n\|(a_{ij})_{ij}\|_A, \ {}_m\|(b_{kl})_{lk}\|_A\}, \max\{ {}_n\|(f(a_{ji}))_{ij}\|_A, \ {}_m\|(f(b_{kl}))_{lk}\|_A\}\} \\ &= \max\{ {}_{n+m}\|(a_{ij})_{ij} \oplus (b_{kl})_{lk}\|_A, \ {}_{n+m}\|(f(a_{ji}))_{ij} \oplus (f(b_{kl}))_{lk}\|_A\} \\ &= \max\{ {}_n\|(a_{ij})\|_B, \ {}_m\|(b_{kl})\|_B\} \end{aligned} $$

and

$$ \begin{aligned} {}_n\|\alpha(a_{ij})\beta\|_B &= \max\{ {}_n\|\alpha(a_{ij})_{ij}\beta\|_A, \ {}_n\|\beta^{\mathsf{T}}(f(a_{ji}))_{ij}\alpha^{\mathsf{T}}\|_A\} \\ &\leq \max\{\|\alpha\| \ {}_n\|(a_{ij})_{ij}\|_A\|\beta\|, \|\beta^{\mathsf{T}}\| \ {}_n\|(f(a_{ji}))_{ij}\|_A\|\alpha^{\mathsf{T}}\|\} \\ &= \|\alpha\|\|\beta\| \max\{ {}_n\|(a_{ij})_{ij}\|_A, \ {}_n\|(f(a_{ji}))_{ij}\|_A\} \end{aligned} $$

Here we use the fact that α and β are scalar matrices. Thus, B is (completely isometrically isomorphic to) an operator space. To prove that B is a pseudoalgebra, observe that

$$\begin{aligned} {}_n\|(a_{ij})(b_{kl})\|_B &= \max\{ {}_n\|(a_{ij})(b_{kl})\|_A,\ {}_n\|f_n((a_{ji})(b_{kl}))\|_A\} \\ &\leq \max\{ {}_n\|(a_{ij})(b_{kl})\|_A, \|f\|_{cb}\, {}_n\|(a_{ij})(b_{kl})\|_A\} \\ &\leq \|f\|_{cb} K\, {}_n\|(a_{ij})\|_A\, {}_n\|(b_{kl})\|_A \\ &\leq \|f\|_{cb} K \cdot \|f\|_{cb} \max\{ {}_n\|(a_{ij})_{ij}\|_A,\ {}_n\|(f(a_{ji}))_{ij}\|_A\}\cdot \\ &\quad \cdot \|f\|_{cb} \max\{ {}_n\|(b_{kl})_{kl}\|_A,\ {}_n\|(f(b_{lk}))_{kl}\|_A\} \\ &= \|f\|_{cb}^3 K\, {}_n\|(a_{ij})\|_B\, {}_n\|(b_{kl})\|_B \end{aligned}$$

Here we use the fact that since $f^2 = 1$ we have that $\|f\|_{cb} \geq 1$.

Since f is *cb*-anti-isomorphism and $f^2 = 1$, we have that

$$\|f\|_{cb}^{-1}\, {}_n\| \cdot \|_A \leq\ {}_n\| \cdot \|_B \leq \|f\|_{cb}\, {}_n\| \cdot \|_A$$

so the algebras A and B are *cb*-isomorphic. Denote this isomorphism by σ. By the construction $(\sigma f \sigma^{-1})^2 = \mathrm{Id}_B$, and so $\sigma f \sigma^{-1}$ is a pseudo-involution. To prove that it is an involution, observe that since $\sigma\colon A \to B$ is a *cb*-isomorphism, every element of $M_n(B)$ may represented as $(b_{ij})_{ij} = \sigma(a_{ij})_{ij}$ for a unique $(a_{ij})_{ij} \in M_n(A)$. Therefore, we have that

$$\begin{aligned} {}_n\|\sigma f \sigma^{-1}(b_{ij})_{ij}\|_B &=\ {}_n\|\sigma f \sigma^{-1}(\sigma((a_{ij})_{ij}))\|_B \\ &=\ {}_n\|\sigma(f(a_{ij}))_{ji}\|_B \\ &= \max\{ {}_n\|(f(a_{ij}))_{ji}\|_A,\ {}_n\|(f^2(a_{ij}))_{ij}\|_A\} \\ &= \max\{ {}_n\|(f(a_{ij}))_{ji}\|_A, {}_n\|(a_{ij})_{ij}\|_A\} \\ &=\ {}_n\|\sigma(a_{ij})_{ij}\|_B \\ &=\ {}_n\|(b_{ij})_{ij}\|_B \end{aligned}$$

and so $\sigma f \sigma^{-1}$ is completely isometric. This last observation finishes the proof. □

Remark 1.2.15. Observe that since f was an anti-isomorphism, we were not able to define σ as just $\sigma\colon a \mapsto a \oplus f(a)$, since in this case $\sigma(ab) = ab \oplus f(ba)$.

The result 1.2.14 gives us only an operator pseudoalgebra with (completely isometric) involution. However, a closer look to the Theorem 1.2.8 lets us extend this result, making B into a (concrete) operator algebra with involution.

In order to do this, we recall the construction from [7]. Let Γ be the set, $n\colon \Gamma \to \mathbb{N}$, $\gamma \mapsto n_\gamma$ be a function. Let Λ be a set of formal symbols (variables) x_{ij}^{γ}, one variable for each $\gamma \in \Gamma$ and each $1 \leq i, j \leq n_\gamma$. Denote by Φ a free associative algebra on Λ. Φ then consists of polynomials in the non-commuting variables with no constant term. Then one defines a norm on $M_n(\Phi)$ by

$$\|(u_{ij})\|_\Lambda := \sup_\pi(\|(\pi(u_{ij}))\|) \tag{1.3}$$

where π goes through all the representations of Φ on a separable Hilbert space satisfying the condition $\|(\pi(x^{\gamma}_{ij}))_{ij}\| \leq 1$ for all γ, where the latter matrix is indexed on rows by $i$and on columns by j for all $1 \leq i, j \leq n_{\gamma}$.

It is then shown in [7] that the map defined above is indeed a norm on $M_n(\Phi)$ and that Φ becomes an operator algebra with respect to these operator norms.

In the proof of the characterization theorem the set Γ is taken to be the collection of $n \times n$ matrices $\gamma = (a_{ij})$ with entries in A such that $\|\gamma\| = \frac{1}{2K}$, where K is a cb-norm of the multiplication in A. Then one takes Λ to be the collection of entries of these matrices $x^{\gamma}_{ij} := a_{ij}$, regarded as formal symbols indexed by γ and i, j, not identifying "equal" entries for different indexes. After that there is defined a map $\theta \colon \Phi \to A$ given by $\theta \colon x^{\gamma}_{ij} \mapsto \gamma_{ij}$ and then extended to general polynomials. It is proved that θ is a completely contractive. One then let $B := \Phi / \ker(\theta)$, which is an operator algebra subject to the quotient operator norm, and is cb-isomorphic to A.

Now let the pseudoalgebra A be involutive. Observe that since the involution on A is completely isometric, we have that ${}_n\|(a_{ij})^*\| = {}_n\|(a_{ij})\|$, and thus $(a_{ij})^* \in \Gamma$. Hence we have that $a^*_{ij} \in \Lambda$. This observation makes us able to define an involution the following way. On Φ we set

$$(x^{\gamma_1}_{i_1 j_1} x^{\gamma_2}_{i_2 j_2} \dots x^{\gamma_k}_{i_k j_k})^* := (x^{\gamma_k}_{i_k j_k})^* (x^{\gamma_{k-1}}_{i_{k-1} j_{k-1}})^* \dots (x^{\gamma_1}_{i_1 j_1})^*$$

on the monomials, and then extend this to the whole Φ. Analogously, on $M_n(\Phi)$ we set $(P_{ij})^* = (P^*_{ji})$.

By the construction we have that $\theta((P_{ij})^*) = \theta((P^*_{ji})_{ij})$. Consequently, let $\pi \colon \Phi \to \mathscr{B}(\mathcal{H})$ be a representation of Φ satisfying the condition $\|(\pi(x^{\gamma}_{ij}))\| \leq 1$ for all $(x^{\gamma}_{ij})_{ij}$. Denote this set by Ξ. We may define a representation $\pi' \colon \Phi \to \mathscr{B}(\mathcal{H})$ by setting $\pi'((P_{ij})^*) := (\pi(P_{ij}))^*$, where the latter involution is given by the one on the $\mathcal{H}^{n_\gamma}$. By the construction, we have that

$${}_{n_\gamma}\|\pi'(x^{\gamma}_{ij})\| = {}_{n_\gamma}\|(\pi((x^{\gamma}_{ij})^*))^*\| = {}_{n_\gamma}\|\pi((x^{\gamma}_{ij})^*)\| \leq 1$$

for all $(x^{\gamma}_{ij})_{ij}$ since $(x^{\gamma}_{ij})^* \in \Gamma$, and so $\pi' \in \Xi$. Therefore we have that

$$\begin{aligned}
\|(P_{ij})\|_{\Lambda} &= \sup_{\pi \in \Xi}(\|(\pi(P_{ij}))\|) \\
&= \sup_{\pi' \in \Xi} \|((\pi'(P_{ij})^*))^*\| \\
&= \sup_{\pi' \in \Xi} \|\pi'(P_{ij})^*\| \\
&= \|(P_{ij})^*\|_{\Lambda}
\end{aligned}$$

Hence we obtain that the map θ respects the involution, and thus the anti-isomorphism induced on B by the involution on Φ preserves the norm.

Combining this observations with Proposition 1.2.14 we have the following

Theorem 1.2.16. *Let A be an operator pseudoalgebra and let f be a pseudo-involution on A. Then there is a cb-isomorphism $\lambda \colon A \to B$, such that the map $\lambda f \lambda^{-1}$ is an (operator algebra) involution on B.*

Proof. Put $\lambda = \theta\sigma$. □

Remark 1.2.17. We may also estimate the *cb*-norm of λ. Indeed, the map σ has the *cb*-norm $\|f\|_{cb}$, and gives us a pseudoalgebra B' with the *cb*-norm of multiplication bounded by $\|f\|_{cb}^3 K$. Assuming that the *cb*-norm of the multiplication map is ≥ 1, we may apply the estimation from [7], which thus gives us that the map θ has a *cb*-norm $\leq \|f\|_{cb}^6 K^2$. Therefore it would hold that $\|\lambda\|_{cb} \leq \|f\|_{cb}^7 K^2$.

1.2.5 Characterization of Operator Modules

Similarly to the cases of operator spaces and operator algebras, there exists a characterization of abstract operator modules. As it should have been expected, this characterization is even more subtle than the one for operator algebras.

Definition 1.2.18. Let A and B be two (possibly abstract) operator algebras and let E be an operator space, which is an A-B-bimodule in algebraic sense. Then E will be called an *abstract cb-A-B-operator bimodule* if the module actions are completely bounded.

Analogously to the case of Hilbert C^*-modules, one may introduce a notion of grading on operator module over a graded operator C^*-algebra.

The following result is due to Blecher:

Theorem 1.2.19 ([9])**.** *Let E be an abstract cb-A-B-operator module. Then there exists a Hilbert space $\mathcal{H}$ and cb-isomorphisms θ, π and ϕ of A, B and E, respectively, into $\mathscr{B}(\mathcal{H})$ such that θ and π are homomorphisms and*

$$\phi(a \cdot \xi \cdot b) = \theta(a)\phi(\xi)\pi(b)$$

for all $a \in A$, $b \in B$ and $\xi \in E$. If, in addition, the algebras are concrete and both module actions are essential, then it is possible to choose a completely isometric θ, π and ψ.

It should be noted, that the second part of the theorem is actually a different result proved using different techniques by Christensen, Effros and Sinclar in [13]. Of course, if $A = \mathbb{C}$ then we may just call E a right B-operator module and similarly for left modules.

1.2.6 Direct Limits of (Abstract) Operator Spaces.

There is a certain notion of direct limit for operator spaces. Namely, let $\{E_\beta\}$ be a family of operator spaces indexed by β in a directed set Δ. Let E_0 be a fixed vector space. Suppose that for all β we have linear maps $\phi_\beta \colon E_0 \to E_\beta$ and $\psi_\beta \colon E_\beta \to E_0$, satisfying the conditions:

1. There exists a positive number C, such that the *cb*-norm of the map $f_{\beta,\gamma} := \phi_\beta \circ \psi_\gamma \colon E_\gamma \to E_\beta$ is $\leq C$ for all β, γ.
2. $\sup_\beta \|\phi_\beta(\xi)\| < \infty$ of all $\xi \in E_0$.
3. for each $\xi \in E_0$, $\psi_\beta(\phi_\beta(\xi)) \to \xi$ in the initial uniform topology.

Here the initial uniform convergence means the following: if $\{\xi_\beta\}$ is a net of elements in E_0, then it converges to an element $\xi \in E_0$ initial uniform if $\sup_\gamma \|\phi_\gamma(\xi_\beta - \xi)\| \to 0$.

In this way we may assign matrix seminorms to E_0, given by

$$_n\|(\xi_{ij})\| = \sup_\beta \|(\phi(\xi_{ij}))\|$$

with reassigning E_0 to denote the quotient of original E_0 by the nullspace of $_1\|\cdot\|$. By the condition (3) above and the triangle inequality, it follows that this supremum actually equals to the limit $\lim_\beta \|(\phi_\beta(\xi_{ij}))\|$.

We have that the metric properties of E_0 are transferred from E_β to E_0, including the local structure on E_0. In particular, one may verify that the conditions of the Theorem 1.2.6 hold, so that the space E_0 may be endowed with the operator space structure.

Definition 1.2.20. The the structure on the space E_0 described above will be called an *inductive limit* operator space structure.

We shall also denote by E the completion of E_0 with respect to the introduced operator norm.

The case of operator modules may be threated analogously. However, the results for abstract operator algebras require additional considerations. We have the following result:

Proposition 1.2.21. *Let A_β be a family of operator pseudoalgebras with multiplication maps μ_β being uniformly completely bounded. Then the direct limit space A is completely boundedly isomorphic to an operator algebra.*

Proof. By Theorem 1.2.6 we already have that A is an operator space. We define the multiplication on A by setting

$$\mu(a,b) := \lim_\beta \psi_\beta(\mu_\beta(\phi_\beta(a), \phi_\beta(b)))$$

where $a, b \in M_n(A)$ and μ_β are the multiplication maps on E_β. One may check directly that this limit exists. Moreover, we have an estimate

$$\begin{aligned}
\|\mu_{a,b}\|_{cb} &= \|\lim_\beta \psi_\beta(\mu_\beta(\phi_\beta(a), \phi_\beta(b)))\|_{cb} \\
&\leq C\|\mu_\beta(\phi_\beta(a), \phi_\beta(b))\|_{cb} \\
&\leq CK\|\phi_\beta(a)\|\|\phi_\beta(b)\| \\
&\leq C^3K\|a\|\|b\|
\end{aligned}$$

and so we may apply the Theorem 1.2.8. □

Remark 1.2.22. It should be noted, that even when one considers concrete operator algebras, their direct limit is in general an operator space that is only *cb*-isomorphic to an operator algebra. In order to apply the Theorem 1.2.9 one has to use some additional information to verify the existence of contractive approximate identity.

An analogous discussion may be found in [9], however there have been imposed many further assumptions that simplified the explanations, but appear to be too restrictive for the theory we are going to develop.

In what following we shall write just ab for $\mu(a,b)$, when it does not lead to a confusion.

We shall also need the following important property of inductive limits.

Theorem 1.2.23 (cf. [9])**.** *Suppose that for all β the operator space E_β is an operator A-module, such that $\mathrm{CB}_A(E_\beta)$ an operator pseudoalgebra, and the multiplication on $\mathrm{CB}_A(E_\beta)$ is uniformly completely bounded for all β. Then $\mathrm{CB}_A(\varinjlim E_\beta)$ is also an operator pseudoalgebra. Moreover, $\mathrm{CB}_A(E_\beta)$ are actual operator algebras and the maps ϕ_β and ψ_β are completely contractive, the resulting homomorphism will also be completely isometric.*

Proof. Let $E = \varinjlim E_\beta$. We define maps $\Phi_\beta \colon \mathrm{CB}_A(E) \to \mathrm{CB}_A(E_\beta)$ and $\Psi_\beta \colon \mathrm{CB}_A(E_\beta) \to \mathrm{CB}_A(E)$ by

$$\Psi_\beta(T) := \psi_\beta T \phi_\beta, \quad \Phi_\beta(S) := \phi_\beta S \psi_\beta$$

These maps are uniformly completely bounded with *cb*-norm $\leq C^2$ and $\Psi_\beta \Phi_\beta(T) \to T$ in point norm topology. Thus, we may check the conditions of Proposition 1.2.21 can be checked locally, with the calculations transferred to the spaces $\mathrm{CB}_A(E_\beta)$. Let T be $n \times n$ matrix of operators from $\mathrm{CB}_A(E)$. By triangle inequality we have that

$$_n\|T\| = \lim_\beta \, _n\|\Phi_\beta(T)\| = \lim_\beta \, _n\|\Psi_\beta \Phi_\beta(T)\|$$

We also have that

$$\begin{aligned} _n\|T_1 T_2\| &= \lim_\beta \, _n\|\psi_\beta \phi_\beta T_1 T_2 \psi_\beta \phi_\beta\| \\ &= \lim_\beta \, _n\|\psi_\beta \phi_\beta T_1 \psi_\beta \phi_\beta T_2 \psi_\beta \phi_\beta\| \\ &= \lim_\beta \, _n\|\Phi_\beta(T_1) \Phi_\beta(T_2)\| \\ &\leq K \lim_\beta \, _n\|\Phi_\beta(T_1)\| \, _n\|\Phi_\beta(T_2)\| \\ &\leq K \, _n\|T_1\| \, _n\|T_1\| \end{aligned}$$

where K is the upper bound for *cb*-norms of multiplications on $\mathrm{CB}(E_\beta, E_\beta)$. This lets us use the Proposition 1.2.21. The proof of the final claim is based on the fact, that with the additional conditions mentioned it is possible to use Theorem 1.2.9 instead of Proposition 1.2.21. □

1.2.7 Haagerup Tensor Product

The notion of Haagerup tensor product has been introduced by Uffe Haagerup in an unpublished paper and then developed by a number of other mathematicians. It is a kind of internal product possessing very important properties, that, however, may only be defined in the case of operator spaces, requiring more information than just a Banach space structure.

Definition 1.2.24. Let X, Y be operator spaces. The *Haagerup norm* on $\mathbb{K} \otimes X \otimes Y$ is defined by

$$\|u\|_h := \inf\left\{\sum_{j=1}^{n} \|x_j\|\|y_j\| \mid u = m(\sum_{j=1}^{n} x_j \otimes y_j),\, x \in \mathbb{K} \otimes X, y \in \mathbb{K} \otimes Y\right\}$$

where

$$\begin{aligned} m\colon \mathbb{K} \otimes X \otimes \mathbb{K} \otimes Y &\to \mathbb{K} \otimes X \otimes Y \\ (a \otimes x) \otimes (b \otimes y) &\mapsto ab \otimes x \otimes y \end{aligned}$$

is the linearization map.

A classical theorem on the Haagerup tensor product reads

Theorem 1.2.25. *The norm on $X \otimes Y$ induced by the Haagerup tensor product equals to*

$$\|u\|_h = \inf\{\|x\|\|y\| \mid x \in X^n,\ y \in Y^n,\ u = \sum_{j=1}^{n} x_j \otimes y_j\}$$

and the completion of $X \otimes Y$ is an operator space.

Definition 1.2.26. The completion of $X \otimes Y$ in the Haagerup norm is called *Haagerup tensor product* and will be denoted as $X \tilde{\otimes} Y$.

An example of Haagerup tensor product is the internal tensor product of C^*-modules.

In fact, the definitions for operator algebras, pseudoalgebras and (bi)modules may be given more naturally in terms of the Haagerup tensor product. For instance, the operator pseudoalgebra is an operator space A which is an algebra, and the multiplication in this algebra induces a completely bounded map $A \otimes A \to A$. Analogously, a (right) operator A-module over an operator pseudoalgebra A is an operator space Y, which is a right module over the algebra A such that the multiplication induces a completely bounded map $Y \tilde{\otimes} A \to Y$. Theorem 1.2.19 guarantees us that in this case there exist a concrete operator space X' and a concrete operator algebra A', such that they are *cb*-isomophic to X and A respectively and the multiplication map is completely contractive.

Now let A be an operator pseudoalgebra, X be a right operator A-module and Y a left operator A-module. Denote by $I_A \subset X \tilde{\otimes} Y$ the closure of linear span of the expressions $(xa \otimes y - x \otimes ay)$.

Definition 1.2.27 ([11])**.** The *module Haagerup tensor product* of X and Y over A is a space

$$X \tilde{\otimes}_A Y := X \tilde{\otimes} Y / I_A$$

equipped with quotient operator norm. It is obviously complete with respect to this norm.

We also have that if X additionally carries a left B_1-module structure and Y carries a right B_2-module structure, then $X \tilde{\otimes}_A Y$ is a B_1-B_2-operator bimodule.

If the operator algebras and the operator modules are graded, one may define a graded Hagerup tensor product in the same way that was used in the subsection 1.1.2.

The Haagerup tensor product also has the following connection to the theory of Hilbert C^*-modules.

Theorem 1.2.28 ([8]). *Let E be a (right) Hilbert C^*-A-module and F be a Hilbert C^*-A,B-bimodule. Then the inner Haagerup tensor product of E and F over A coincides with the spatial tensor product $E \tilde{\otimes}_A F$.*

This result resolves the ambiguity in the notation of spatial and Haagerup tensor products for Hilbert C^*-modules.

Basing on this result, we obtain a description of compact operators on Hilbert C^*-modules that may then be generalized for further purposes. Namely for a (right) C^*-module E we define an *dual module*

$$E^* := \{\xi^* \mid \xi \in E\}$$

with the structure of left A-module given by

$$a\xi^* := (\xi a)^*$$

and a C^*-module structure defined as

$$(\xi_1^*, \xi_2^*) \mapsto \langle \xi_1, \xi_2 \rangle$$

We have the following result.

Theorem 1.2.29 ([8]). *Let E be a Hilbert C^*-A-module and F be Hilbert C^*-B-module, ccrying an essential representation on A. Then there is a completely isometric isomorphism*

$$\mathbb{K}_B(E \tilde{\otimes} F) \xrightarrow{\sim} E \tilde{\otimes}_A \mathbb{K}_B(F) \tilde{\otimes}_A E^*$$

In particular, $\mathbb{K}_B(E) = E \tilde{\otimes} E^*$.

1.2.8 Rigged and Almost Rigged Modules

The works of Blecher, particularly [8] and [9] have produced an insightful view of the Hilbert C^*-modules, providing the way for many possible generalizations. The main observation was the so called "approximate projectivity" of Hilbert C^*-modules. Namely, for any Hilbert C^*-A-module E there exists an approximate unit $\{u_\alpha\}_{\alpha \in \Lambda}$, $u_\alpha \in \mathrm{Fin}_A(E)$ for the algebra $\mathbb{K}_A(E)$. Replacing, when needed, u_α with $u_\alpha u_\alpha^*$, we may assume

$$u_\alpha = \sum_{j=1}^{k_\alpha} x_j^\alpha \otimes x_j^\alpha$$

For each α one the constructs operators $\phi_\alpha \in \mathbb{K}_A(E, A^{n_\alpha})$, defined by

$$\phi_\alpha \colon \xi \mapsto \sum_{j=1}^{k_\alpha} e_j \langle x_j^\alpha, \xi \rangle \tag{1.4}$$

where e_j denote the standard basis of A^{k_α}. We may also construct the adjoint map

$$\phi_\alpha^* \colon x \mapsto \sum_{j=1}^{k_\alpha} x_j^\alpha \langle e_j, x \rangle \tag{1.5}$$

and we have that $\phi_\alpha^* \circ \phi_\alpha \to \mathrm{Id}_E$. It was proved in [8] that this structure defines the Hilbert C^*-module structure completely. In fact, Blecher have proved a more general statement

Theorem 1.2.30. *[9] Let A be a C^*-algebra and let E be a Banach (operator) space, which is also a right (operator) module over A. Then E is completely isometrically isomorphic to a countably generated Hilbert C^*-module if and only if there exists a sequence k_α of positive integers and contractive module maps*

$$\phi_\alpha\colon E \to A^{k_\alpha}, \quad \psi_\alpha\colon A^{k_\alpha} \to E$$

such that $\psi_\alpha \circ \phi_\alpha$ converges pointwise to the identity on E. The inner product in this case is given by

$$\langle \xi, \eta \rangle = \lim_{\alpha\to\infty} \langle \phi_\alpha(\xi), \phi_\alpha(\eta) \rangle$$

This result gave rise to the following concept, generalizing Hilbert C^*-modules.

Definition 1.2.31 ([9])**.** Let $\mathcal{A}$ be an operator algebra and E be an operator A-module. The module E will be called *rigged*, if there exist a net of maps $\phi_\alpha\colon E \to A^{n_\alpha}$ and $\psi_\alpha\colon A^{n_\alpha} \to E$ such that for all α, β

1. the maps ϕ_α and ψ_α are completely contractive;
2. $\psi_\alpha \phi_\alpha \to \mathrm{Id}_E$ strongly on E;
3. $\|\phi_\alpha \psi_\beta \phi_\beta - \phi_\alpha\|_{cb} \xrightarrow{\beta} 0$;
4. The maps ψ_α are A-essential.

Rigged modules possess many important properties which make them very similar to the Hilbert C^*-modules. Here we present two theorems indicating this analogy, which would be used in the subsequent constructions:

Theorem 1.2.32 ([9])**.** *Let A be a C^*-algebra. Then E is a rigged module over A if and only if E is a Hilbert C^*-modules over A.*

Theorem 1.2.33 ([9])**.** *Let A and B be operator algebras, E be a rigged A-module, Y be a rigged B-module and $\pi\colon A \to \mathrm{End}_B(Y)$ be a completely contractive essential morphism. Then the Haagerup tensor product $E \tilde{\otimes}_A Y$ is a rigged B-module.*

However, rigged module do not suffice for the unbounded KK-theory we are going to develop in Chapter 2. Therefore we are going to apply a more general notion of almost rigged modules. We postpone an example, illustrating this choice, to the next chapter.

Definition 1.2.34. An operator module will be called *almost rigged* if it satisfies all the conditions of 1.2.31, with the exception that the property 1 is replaced by

1′. there is a positive constant C such that $\|\phi_\alpha\|_{cb} \le C$ and $\|\psi_\alpha\|_{cb} \le C$ for all α.

Observe that in case when $C \leq 1$ an almost rigged module is a genuine rigged module.

The almost rigged modules have been studied by Blecher, but as far as the author knows the paper devoted to them remains unpublished. Perhaps, these kind of modules haven't had the properties, that were needed by Blecher in his own research. However, they do have the ones that would be utile in the unbounded KK-theory we are going to study in the Chapter 2. These properties, that, in fact, are the analogues of the ones found by Blecher, have been studied my Mesland in [28], and we are going to follow his approach in this part.

We also emphasize here that since in our applications we shall primarily need countably generated modules, all the modules that we are going to consider are supposed to be countably generated, or, more precisely, the net of the indexes $\{\alpha\}$ for ϕ_α and ψ_α has a countable set of elements.

For an almost rigged module E we define the *dual almost rigged module* E^* as

$$E^* := \{\eta^* \in \mathrm{CB}_A(E, A) \mid \eta^* \circ \psi_\alpha \circ \phi_\alpha \to \eta^*\}$$

and the space of *A-compact operators on E* $\mathbb{K}_A(E)$ as a closure of the set of finite rank operators $T_{\xi,\eta^*}(\zeta) = \xi \cdot \eta^*(\zeta)$ for $\xi, \zeta \in E$ and $\eta^* \in E^*$. Thus, by the construction we have that the space $\mathbb{K}_A(E)$ is cb-isomorphic to $E \tilde{\otimes} E^*$.

Proposition 1.2.35. *For the right almost rigged A-module E the module E^* indeed has a left almost rigged A-module structure.*

Proof. First of all, E^* is a left operator A-module, with the module structure given by $(a\eta^*)\xi = a(\eta^*\xi)$. To impose the almost rigged structure, we first let $y^j_\alpha = \psi_\alpha(e_j) \in E$, where e_i is a standard basis on A^{k_α}, so that $\psi_\alpha(\sum_{j=1}^{k_\alpha} e_j a_j) = \sum_{j=1}^{k_\alpha} y^j_\alpha a_j$, and also denote by $f^j_\alpha \in \mathrm{Hom}^c(E, A^{k_\alpha})$ the elements with the property that $(f^j_\alpha(\xi))_l = 0$ for $l \neq j$ and $\phi_\alpha(\xi) = \sum_{j=1}^{k_\alpha} f^j_\alpha(\xi)$. The latter elements exist since the maps ϕ_α are linear. Subsequently, we define the structural maps $\psi^*_\alpha \colon E^* \to (A^{k_\alpha})^\intercal$ and $\phi^*_\alpha \colon (A^{k_\alpha})^\intercal \to E^*$ for E^* by

$$\psi^*_\alpha(\eta^*) := \sum_{j=1}^{k_\alpha} \eta^*(y^j_\alpha) e^*_j, \quad \phi^*_\alpha(\sum_{j=1}^{k_\alpha} a_j e^*_j) := \sum_{j=1}^{k_\alpha} a_j f^j_\alpha$$

To see that these maps indeed define an almost rigged structure on E, we first observe that the maps $y^j_\alpha \colon y^j_\alpha(\eta^*) := \eta^*(y^j_\alpha)$ are have by the construction a cb-norm $\leq C$, and thus so does ψ^*_α. As to ϕ^*_α, by the construction of f^j_α we have that

$$\| \sum_{j=1}^{k_\alpha} a_j f^j_\alpha \|_{cb} \leq \max_{j=1,\dots,k_\alpha} \|a_j f^j_\alpha\|_{cb} \leq C \max_{j=1,\dots,k_\alpha} \|a_j\|$$

so that we may conclude that the cb-norm of ϕ^*_α is $\leq C$ as well. The other properties follow automatically by the definition of E^*. □

This characterization of E^* gives us a notion of adjointable operators on almost rigged modules.

Definition 1.2.36. Let E, F be almost rigged modules over an operator A and let $T \in \mathrm{CB}(E,F)$. Then T will be called *adjointable* if there exists an operator $T^*\colon F^* \to E^*$ such that

$$\langle T\xi, \eta^* \rangle = \langle \xi, T^*\eta^* \rangle$$

for all $\xi \in E$ and $\eta^* \in F^*$. The set of all adjointable operators $T\colon E \to F$ will be then denoted $\mathrm{CB}^*(E,F)$.

Almost rigged modules satisfy the direct limit property mentioned in [9], with the difference that instead of complete contractiveness of we require the maps to be completely bounded with uniform upper bound. The transition maps $t_{\alpha\beta}\colon A^{k_\beta} \to A^{k_\alpha}$ are given by $t_{\alpha\beta} := \phi_\alpha\psi_\beta$. This endows the almost rigged modules with the following universal property:

Proposition 1.2.37. *Let E be an almost rigged module over an operator algebra A and let $g_\alpha\colon A^{n_\alpha} \to W$ be completely bounded module maps with a uniformly bounded cb-norms for some operator space W, such that $g_\alpha t_\alpha\beta \to g_\beta$ strongly. Then there is a unique completely bounded morphism $g\colon E \to W$ for which $g_\beta = g\phi_\beta$.*

Proof. Define $g(\psi_\gamma(x)) = g_\gamma(x)$ for $x \in A^{k_\gamma}$. We observe that

$$g_\gamma(x) = \lim_\beta g_\beta t_{\gamma\beta}(x) = \lim_\beta g_\beta\phi_\beta\psi_\gamma(x)$$

The latter expression is well defined for any x, since the morphisms of g_β, ϕ_β and ψ_γ are completely bounded with *cb*-norm $\leq C$ for some $C \in \mathbb{R}$ independent of β, we have that g is also completely bounded. The uniqueness of this morphism may be checked in the standard way. □

The completely bounded operators on almost rigged modules satisfy the following properties:

Theorem 1.2.38 (cf. [9])**.** *Let E be a right almost rigged A-module over an operator algebra A. Then*

1. *The space $\mathrm{CB}_A(E)$ is completely isometrically isomorphic to an operator algebra.*
2. *The algebra $\mathbb{K}_A(E)$ is a left ideal in $\mathrm{CB}_A(E)$, with a uniformly bounded approximate unit given by $t_{\beta\beta}$.*

Proof. Indeed, for (1) we have by [9] that $\mathrm{CB}(A^n)$ is completely isometrically isomorphic to an operator algebra. Therefore we may apply the Theorem 1.2.23. By the construction, $\mathrm{CB}_A(E)$ is an operator 1-algebra. It also contains a unit given by Id_E. Therefore, by Theorem 1.2.9 it is isomorphic to an operator algebra completely isometrically.

For (2) we observe that for $S \in \mathrm{CB}_A(E)$ one has $ST_{\xi,\eta^*} = T_{S_\xi,\eta^*}$, so $\mathbb{K}_A(E)$ is indeed a left ideal in $\mathrm{CB}_A(E)$. Now for $\xi \in E$ and $\eta^* \in E^*$ we have that $t_{\beta\beta}T_{\xi,\eta^*} = T_{t_{\beta\beta}\xi,\eta^*}$ and $T_{\xi,\eta^*}t_{\beta\beta} = T_{\xi,t^*_{\beta\beta}\eta^*}$. By the definition 1.2.34 and the fact that $\|T_{\xi,\eta^*}\|_{cb} \leq C\|\xi\|\|\eta^*\|_{cb}$ for some positive constant C, these both operators converge to T_{ξ,η^*}. Therefore $t_{\beta\beta}$ is a bounded approximate unit for $\mathbb{K}_A(E)$.

□

There is another aspect that will be useful for our further purposes. It is the question, whether the rigged module my be stabilized, i.e., whether it is a direct summand of $\mathcal{H}_A$. In general, this is not true even for countably generated almost rigged module. However, there is an additional condition on rigged modules that settles this problem.

Definition 1.2.39 ([11]). An operator module E over an operator algebra A is said to have a (P)*-quasi-unit* if there are completely bounded maps $\phi\colon E \to \mathcal{H}_A$ and $\psi\colon \mathcal{H}_A \to E$ such that $\psi \circ \phi = \mathrm{Id}_E$ and ψ is finitely A essential.

The property of being A-essential means that the restrictions of ψ to $\mathcal{A}^{(n)}$ are right A-essential.

The modules with (P)-quasi-unit are obviously almost rigged (take $\phi_n = p_n\phi$ an $\psi_n = \psi p_n$, where p_n are standard projections $\mathcal{H}_A \to A^n$). These operator modules have the properties that relate them to countably generated Hilbert C^*-modules.

Theorem 1.2.40 ([9]). *The modules with P-quasi units with completely contractive ϕ and ψ over C^*-algebras are exactly countably generated Hilbert C^*-modules.*

These modules do satisfy the stabilization property.

Theorem 1.2.41 ([11]). *Let E be an operator module with (P)-quasi-unit over an operator algebra A. Then*

- *There exists a complemented submodule N in $\mathcal{H}_A$ such that there is a cb-isomorphism $\mathcal{H}_A \approxeq E \oplus N$.*
- *There is a cb-isomorphism $E \oplus \mathcal{H}_A \approxeq \mathcal{H}_A$*
- *There is a cb-isomorphism $\mathcal{H}_A \oplus \mathcal{H}_E \approxeq \mathcal{H}_A$*

Because of this result, we shall also call the modules with P-quasi unit *cb-stabilizable*.

1.2.9 Haagerup Tensor Product of Almost Rigged Modules

In this we are going to show that the Haagerup tensor product of two almost rigged modules is again an almost rigged module.

As well as for C^*-algebras and Hilbert C^*-modules, there is a notion of essential homomorphisms for operator algebras. However, since the operator algebra homomorphisms should not necessarily be contractive as in C^*-algebra case, the definition is somewhat more involved. Namely, we have:

Definition 1.2.42 ([9]). Let A and B be operator algebras and $f\colon A \to \mathscr{M}(B)$ be a completely bounded homomorphism. Then f is called *essential* if one of the following equivalent conditions hold:

1. $\{f(e_\alpha)\}$ converges strictly to the identity in $\mathscr{M}(B)$ for any bounded approximate identity $\{e_\alpha\}$ of A;

2. $\{f(u_\alpha)\}$ converges strictly to the identity in $\mathscr{M}(B)$ for some norm-bounded net $\{u_\alpha\}$ in A;

3. any element of B may be expressed as a product $f(a)b$ and also as $b'f(a')$ for some $a, a' \in A$ and $b, b' \in B$

4. f has a unique completely bounded extension $f' \colon \mathscr{M}(A) \to \mathscr{M}(B)$, such that, when restricted to bounded subsets, f' is continuous with respect to strict topologies.

It is proved in [9][Thm 6.2] that these four conditions are equivalent. It is also shown there that the extension f' has the same *cb*-norm as f.

We need the notion of essential homomorphism for the following result:

Lemma 1.2.43. *Let A, B be operator algebras and $f \colon A \to \mathscr{M}(B)$ be an essential homomorphism. Then $B \approxeq A \tilde{\otimes}_A B$ completely boundedly.*

Proof. Let A_1 be an algebra consisting (algebraically) of the same elements as A. We make it into an operator algebra via a representation

$$\sigma \colon a \mapsto \begin{pmatrix} a & 0 \\ 0 & f(a) \end{pmatrix}$$

Then, by the construction, $\sigma \colon A \to A_1$ is a *cb*-isomorphism, $f\sigma^{-1}$ is completely contractive and $A_1 \to \mathscr{M}(B)$ is essential. It is proved in [9] that the assertion of the lemma holds for completely contractive essential morphisms. Therefore $A_1 \tilde{\otimes}_{f\sigma^{-1}} B \approxeq B$. Finally, since σ is a completely bounded isomorphism, there is *cb*-isomorphism between $A \tilde{\otimes}_f B$ and $A_1 \tilde{\otimes}_{f\sigma^{-1}} B$. □

Now we are ready to prove that almost rigged modules remain almost rigged under the base change.

Theorem 1.2.44 (cf. [9])**.** *Let E be a right almost rigged operator A-module and let $A \to B$ be an essential homomorphism of operator algebras. Then $E \tilde{\otimes}_A B$ is a right almost rigged B-module. Moreover, $\mathbb{K}_B(E \tilde{\otimes}_A B) \approxeq E \tilde{\otimes}_A B \tilde{\otimes}_A E^*$ (completely bounded isomorphism).*

Proof. It was indicated in [11] that this result should hold, but there has not been given a direct proof. The author decided to give a precise proof of this fact. We follow [9, Thm. 6.4]. By Lemma 1.2.43 we have that $A \tilde{\otimes}_A B = B$, and hence

$$A^n \tilde{\otimes}_A B \approxeq \mathbb{C}^n \tilde{\otimes} A \tilde{\otimes}_A B \approxeq B^n$$

where the isomorphisms are completely bounded.

We choose a bounded approximate unit $\{\varepsilon_\alpha\}$ for B, and write

$$\begin{aligned} L_\lambda \colon B &\to B \\ b &\mapsto \varepsilon_\lambda b \end{aligned}$$

Let ϕ_α and ψ_α define the structure of almost rigged module on E. We denote by $\phi'_{\alpha,\lambda}\colon \mathcal{E}\tilde{\otimes}_A B \to B^{n_\alpha}$ and $\psi'_{\alpha,\lambda}\colon B^{n_\alpha} \to \mathcal{E}\tilde{\otimes}_A B$ the morphisms

$$\phi'_{\alpha,\lambda} = L_\lambda \circ (\phi_\alpha \tilde{\otimes} \mathrm{Id}_B),\ \psi'_{\alpha,\lambda} = L_\lambda \circ (\psi_\alpha \tilde{\otimes} \mathrm{Id}_B)$$

Then the requirements (1) and (4) of the Definition 1.2.34 are obviously satisfied. For the assertions (2) and (3), the only recall that we are dealing with countable nets, therefore we may re-index the indexes (α, λ) to get a countable indexation ω for ϕ' and ψ'. Thus E becomes an almost rigged module over A.

As for the compact operators, we first observe that by definition of dual rigged module we have that $(E\tilde{\otimes}_A B)^* \approxeq B\tilde{\otimes}_A E^*$ *cb*-isomorphically. Therefore we have that

$$\mathbb{K}_B(E) \approxeq (E\tilde{\otimes}_A B)\tilde{\otimes}(E\tilde{\otimes}_A B)^* \approxeq E\tilde{\otimes}_A (B\tilde{\otimes} B)\tilde{\otimes}_A E^* \approxeq E\tilde{\otimes}_A B\tilde{\otimes}_A E^*$$

□

Thus we have shown that the Haagerup tensor product is functorial on almost rigged modules. Moreover, this result may be generalized to yield the product of almost rigged modules

Theorem 1.2.45 (cf. [9])**.** *Let E be an almost rigged A-module, F be an almost rigged B-module and $\pi\colon A \to \mathbb{B}(F)$ be an essential map. Then $E\tilde{\otimes}_A F$ is an almost rigged B-module. We also have that $\mathbb{K}_B(E\tilde{\otimes}_A F) \approxeq F\tilde{\otimes}_A \mathbb{K}_A(E)\tilde{\otimes}_A F$.*

Proof. Let ϕ_α, ψ_α and ϕ'_λ, ψ'_λ be factorization maps for E and F respectively. We construct the maps $\Phi_{\alpha,\lambda}\colon E\tilde{\otimes}_A F \to B^{n_\alpha m_\lambda}$ and $\Psi_{\alpha,\lambda}\colon B^{n_\alpha m_\lambda} \to E\tilde{\otimes}_A F$ by setting

$$\Phi_{\alpha,\lambda} := \mathrm{diag}_{m_\lambda}(\phi'_\lambda)\circ(\phi_\alpha \otimes \mathrm{Id}_F);\ \Psi_{\alpha,\lambda} := (\psi_\alpha \otimes \mathrm{Id}_F)\circ \mathrm{diag}_{m_\lambda}(\psi'_\lambda)$$

where we use the Theorem 1.2.44 to establish the *cb*-isomorphism $A^{n_\alpha}\tilde{\otimes}_A F \approxeq F$. The maps $\Phi_{\alpha,\lambda}$ and $\Psi_{\alpha,\lambda}$ then endow the module $E\tilde{\otimes}_A F$ with the struture of rigged B-module. Indeed, since ϕ_α, ψ_α and $\phi'_\lambda, \psi'_\lambda$ are uniformly completely bounded, so are, by the construction, the maps $\Phi_{\alpha,\lambda}$ and $\Psi_{\alpha,\lambda}$. Therefore the condition (1) of the Definition 1.2.34 is fulfilled. Since ψ'_λ is B-essential, we have that the same holds for $\mathrm{diag}(\psi'_\lambda)$. Now, since the map ψ_α is A-essential and Id_F is obviously B-essential, we obtain that the map $\psi_\alpha \otimes \mathrm{Id}_F$ is B-essential. Therefore $\Psi_{\alpha,\lambda}$ is B-essential. The conditions (2) and (3) may also be checked directly if we make a reordering as in Theorem 1.2.44.

Finally, using the same factorization maps we may show that $(E\tilde{\otimes}_A F)^* \approxeq F^*\tilde{\otimes}_A E^*$, and so

$$\mathbb{K}_B(E\tilde{\otimes}_A F) = E\tilde{\otimes}_A F\tilde{\otimes} F^*\tilde{\otimes}_A E^* = E\tilde{\otimes}_A \mathbb{K}_B(F)\tilde{\otimes}_A E^*$$

□

Finally we obtain the corollary for the *cb*-stabilizable modules.

Corollary 1.2.46. *Let E and F be cb-stablizable A and B -modules respectively and suppose that there are essential maps $f\colon A \to \mathbb{B}_B(E)$ and $g\colon A \to \mathbb{B}_B(\mathcal{H}_B)$. Then $E\tilde{\otimes}_A F$ is cb-stabilizable.*

Proof. Indeed, we write down

$$(E \oplus \mathcal{H}_A) \tilde{\otimes}_A (F \oplus \mathcal{H}_B) = (E \tilde{\otimes}_A F) \oplus (E \tilde{\otimes}_A \mathcal{H}_B \oplus \mathcal{H}_A \tilde{\otimes}_A F \oplus \mathcal{H}_A \tilde{\otimes}_A \mathcal{H}_B)$$

Since there is an essential map $g\colon A \to \mathbb{B}_B(\mathcal{H}_B)$ we establish a *cb*-isomorphism $\mathcal{H}_A \tilde{\otimes}_A \mathcal{H}_B \approxeq \mathcal{H}_B$. Analogously, we have that $\mathcal{H}_A \tilde{\otimes}_A F = \mathcal{H}_F$ and $E \tilde{\otimes}_A \mathcal{H}_B = \mathcal{H}_{E\tilde{\otimes}_A B}$. Since F is *cb*-stabilizable, we have that $\mathcal{H}_F \oplus \mathcal{H}_A \approxeq \mathcal{H}_A$. Now,

$$\begin{aligned} E \tilde{\otimes}_A B \oplus \mathcal{H}_B &\approxeq E \tilde{\otimes}_A B \oplus \mathcal{H}_A \tilde{\otimes}_A \mathcal{H}_B \\ &\approxeq (E \oplus \mathcal{H}_A) \tilde{\otimes}_A \mathcal{H}_B \\ &\approxeq \mathcal{H}_A \tilde{\otimes}_A \mathcal{H}_B \\ &\approxeq \mathcal{H}_B \end{aligned}$$

Thus we may write

$$E \tilde{\otimes}_A F \oplus \mathcal{H}_B \approxeq (E \oplus \mathcal{H}_A) \tilde{\otimes}_A (F \oplus \mathcal{H}_B)$$

But, again, by definition we have that $E \oplus \mathcal{H}_A \approxeq \mathcal{H}_A$ and $F \oplus \mathcal{H}_B \approxeq \mathcal{H}_B$, and so

$$E \tilde{\otimes}_A F \oplus \mathcal{H}_B \approxeq \mathcal{H}_A \tilde{\otimes}_A \mathcal{H}_B \approxeq \mathcal{H}_B$$

QED. □

Remark 1.2.47. The claim for A to have an essential representation on $\mathcal{H}_B$ is in fact not too restrictive. For instance, it is enough for A to have an essential representation on a separable Hilbert space $\mathcal{H}$. We recall that since A is an operator algebra it is already a subalgebra of $\mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, so that the task is to find a separable $\mathcal{H}$ Hilbert subspace of $\mathcal{H}$, stable under the action on of A. The essential action of A on $\mathcal{H}_B = \mathcal{H} \tilde{\otimes} B$ will then be given by the map $a \mapsto a \otimes \mathrm{Id}_B$.

If A is a C^*-algebra, it is enough for A to have an essential representation on any full countably generated Hilbert C^*-B-module E. Indeed, by Theorem 1.1.3 $\mathcal{H}_E \approxeq \mathcal{H}_A$, and therefore there is an essential *cb*-representation of E on $\mathcal{H}_A$.

1.2.10 Stuffed Modules

Starting from this subsection, all the algebras are supposed to have a bounded approximate unit, and the morphisms are supposed to be essential.

The notion of stuffed modules is based on the notion of smooth modules introduced in [28]. In a sense we are giving the Mesland's notion its own right for existence, not necessarily binded to the context of Sobolev chains they were introduced in [28]. The techniques would be very similar to the ones in [28], although there will be a considerable difference due to the fact that we are working with more general objects.

Definition 1.2.48. Let A be a C^*-algebra, $\mathcal{A}$ be a pre-C^*-subalgebra of A and $\pi\colon \mathcal{A} \to B(\mathcal{H})$ be monomorphism endowing $\mathcal{A}$ with an operator algebra structure. Let also the algebras $M_k(\mathcal{A})$ be complete in the norm $\|\cdot\|_{\mathcal{A}} = \|\pi(\cdot)\|$ and the inclusion map $\mathcal{A} \hookrightarrow A$ be completely bounded with respect to the operator structure on $\mathcal{A}$ defined by the map π.

Finally, let E be Hilbert C^*-module over A. The *pre-stuffed $\mathcal{A}$-module structure* on E is then given by an approximate unit

$$u_\alpha := \sum_{j=1}^{k_\alpha} x_j^\alpha \otimes x_j^\alpha \in \mathrm{Fin}_A E$$

with x_j^α being homogeneous elements such that the matrices $(\langle x_j^\alpha, x_k^\beta \rangle) \in M_n(A)$ for each n and

$$\|(\langle x_j^\alpha, x_k^\beta \rangle)\|_{\mathcal{A}} \leq C$$

For short we shall call the operator algebras mentioned in the above definition as *operator pre-C^*-algebras*. Also, in the following we are going to identify the notation for pre-C^*-algebra $\mathcal{A}$ with the operator algebra given by $\pi(\mathcal{A})$. We shall also write a^* for the $\pi(a^*)$.

Proposition 1.2.49. *In the conditions of definition 1.2.48 the module*

$$\mathcal{E} := \{\xi \in E \mid \langle x_j^\alpha, \quad \sup_k \| \sum_{i=1}^{k_\alpha} e_i \langle x_j^\alpha, \xi \rangle \|_{\mathcal{A}} < \infty\}$$

is dense in E and is an almost rigged operator module over $\mathcal{A}$.

Proof. We recall the discussion over the introduction of rigged modules. We define set the maps

$$\psi_\alpha \colon \mathcal{A}^{m_\alpha} \to \mathcal{E}, \quad \phi_\alpha \colon \mathcal{E} \to \mathcal{A}^{m_\alpha}$$

and define matrix norms on $\mathcal{E}$ by

$$\|(\xi_{ij})\|_{\mathcal{E}} := \sup_k \|(\phi_k(\xi_{ij}))\|_{\mathcal{A}}$$

It is straightforward to prove that these matrix norms satisfy the conditions of $\mathcal{L}^\infty$-matricially normed space, so that $\mathcal{E}$ is completely isometrically isomorphic to an operator space. One may also check directly that by definition the maps ϕ_k and ψ_k endow $\mathcal{E}$ with almost rigged module structure over $\mathcal{A}$. To show that $\mathcal{E}$ is dense n E it suffices for us to prove that x_j^β lie in $\mathcal{E}$, since they for a generating system for E. Indeed,

$$\begin{aligned}
\|x_j^\beta\|_{\mathcal{E}}^2 &= \sup_\alpha \|\phi_\alpha(x_j^\beta)\|_{\mathcal{A}}^2 \\
&= \sup_\alpha \| \sum_{i=1}^{k_\alpha} e_i \langle x_i^\alpha, x_j^\beta \rangle \|_{\mathcal{A}}^2 \\
&= \sup_\alpha \| \sum_{i=1}^{k_\alpha} \pi(\langle x_i^\alpha, x_j^\beta \rangle)^* \pi(\langle x_i^\alpha, x_j^\beta \rangle) \| \\
&\leq \|(\pi(\langle x_i^\alpha, x_j^\beta \rangle))_{ij}\|^2 \\
&= \|(\langle x_i^\alpha, x_j^\beta \rangle)_{ij}\|_{\mathcal{A}}^2 \\
&\leq C
\end{aligned}$$

Therefore we ave indeed proved that $x_j^\beta \in \mathcal{E}$, and hence $\mathcal{E}$ is dense in E.

□

Definition 1.2.50. The pre-stuffed A-module structure will be called *stuffed $\mathcal{A}$-module structure* if the module $\mathcal{E}$ is *cb*-stabilizable as an almost rigged module over $\mathcal{A}$.

There is a well defined inner product on stuffed modules, which is inherited form the inner product on Hilbert C^*-modules. To prove this, we first have to show that the stuffed modules are self-dual.

First of all, for any right almost rigged $\mathcal{A}$-module E there is a canonically associated left almost rigged $\mathcal{A}$-module $\tilde{\mathcal{E}}$ defined as

$$\tilde{\mathcal{E}} := \{\overline{\xi} \mid \xi \in \mathcal{E},\ a\overline{\xi} := \overline{\xi a}\}$$

The structure of pre-stuffed module on $\tilde{E}$ is then given by the completely isometric anti-isomorphism

$$\begin{array}{rcl} \mathcal{A}^{k_\alpha} & \to & (\mathcal{A}^{k_\alpha})^\intercal \\ (a_j) & \mapsto & (a_j^*)^\intercal \end{array}$$

and the structural maps are given by

$$\begin{array}{rcl} \tilde{\phi}_\alpha(\overline{\xi}) & := & ((\phi_\alpha(\xi))_j^*)^\intercal \\ \tilde{\psi}_\alpha((a_j)^\intercal) & := & \overline{\psi_\alpha((a^*)_j)} \end{array}$$

It is straightforward to check that these maps make $\tilde{\mathcal{E}}$ into an almost rigged module. It follows from definition that $\tilde{\mathcal{E}}$ is stabilizable whenever $\mathcal{E}$ is stabilizable.

Lemma 1.2.51 ((cf. [28])). *Let $\mathcal{E}$ be a stuffed module over a pre-C^*-algebra $\mathcal{A}$. Then there is a cb-isomorphism of almost rigged modules $\mathcal{E}^* \approxeq \tilde{\mathcal{E}}$ given by the restriction of the inner product pairing on E.*

Proof. Obviously, we have an injection $\tilde{\mathcal{E}} \to \mathcal{E}^*$, $\overline{\xi} \mapsto \xi^*$, defined by the restriction of the inner product on E. It suffices for us to construct a completely bounded inverse map. We define

$$\begin{aligned} g_\beta \colon (\mathcal{A}^{k_\beta})^\intercal &\to \tilde{\mathcal{E}} \\ (a_j)^\intercal &\mapsto \sum_{j=1}^{k_\beta} a_j \overline{x}_j^\beta \end{aligned}$$

We would like to apply the direct limit property of the almost rigged modules. By definition we already have that $\|g_\beta\|_{cb} \leq c$ for some positive $c \in \mathbb{R}$. Thus we need to check that

$g_\beta \psi_\beta^* \phi_\alpha^* \to g$, where ψ_β^*, ϕ_α^* defined as in Proposition 1.2.35. Indeed,

$$\begin{aligned}
g_\beta \psi_\beta^* \phi_\alpha^* (a_i)^\intercal &= g_\beta \psi_\beta^* (\sum_{i=1}^{k_\alpha} a_i x_i^{\alpha *}) \\
&= g_\beta (\sum_{i=1}^{k_\alpha} a_i \langle x_i^\alpha, x_j^\beta \rangle)_j^\intercal \\
&= \sum_{j=1}^{k_\beta} \left\langle \sum_{i=1}^{k_\alpha} x_\alpha^i a_i^*, x_j^\beta \right\rangle \overline{x}_j^\beta \\
&= \overline{x_j^\beta \sum_{j=1}^{k_\beta} \left\langle \sum_{i=1}^{k_\alpha} x_j^\beta, x_i^\alpha a_i^* \right\rangle} \\
&\xrightarrow{\beta} \overline{\sum_{i=1}^{k_\alpha} x_i^\alpha a_i^*} \\
&= \sum_{i=1}^{k_\alpha} a_i \overline{x}_i^\alpha \\
&= g_\alpha (a_i)^\intercal
\end{aligned}$$

Therefore, by Proposition 1.2.37 we are able to construct the induced map $g\colon \mathcal{E}^* \to \tilde{\mathcal{E}}$. By the construction, we have that $g(x_i^{\alpha *}) = \overline{x}_i^\alpha$, and hence $g(\xi^*) = \overline{\xi}$ for an arbitrary $\xi \in E$. Thus g is a left inverse for $\overline{\xi} \mapsto \xi^*$. Since $x_i^{\alpha *}$ generate $\mathcal{E}^*$ it is also a right inverse. Thus, we have established the completely bounded isomorphism $\mathcal{E}^* \approx \tilde{\mathcal{E}}$. □

In effect, the Lemma 1.2.51 tells us, that there is a nondegenerate $\mathcal{A}$-valued inner product pairing on $\mathcal{E}$ induced by the A-inner product on E. In this sense the stuffed module E over the pre-C^*-algebra A may be regarded as a pre-C^*-module.

The next proposition will play an important role in the upcoming sections.

Proposition 1.2.52 (cf. [28])**.** *Let A be a C^*-algebra and $\mathcal{A}_1$, $\mathcal{A}_2$ be two operator pre-C^*-subalgebras of A, such that there is inclusion $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$, which is completely bounded. Let E be a countably generated Hilbert C^*-module over A and $\{u_k\}$ be an approximate unit on $\mathbb{K}_A(E)$ defining the structure of stuffed module on E for both $\mathcal{A}_1$ and $\mathcal{A}_2$, with $\mathcal{E}_1$ and $\mathcal{E}_2$ be the operator modules corresponding to $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Then there is a cb-isomorphism $\mathcal{E}_1 \tilde{\otimes}_{\mathcal{A}_1} \mathcal{A}_2 \approxeq \mathcal{E}_2$.*

Proof. The proposition was contained as a part of a [28, Thm. 4.4.2] with the specific types of operator algebras.

Indeed, the isomorphism is implemented via the multiplication map

$$\begin{aligned}
m\colon \mathcal{E}_1 \tilde{\otimes}_{\mathcal{A}_1} \mathcal{A}_2 &\to \mathcal{E}_2 \\
\xi \otimes a &\mapsto \xi a
\end{aligned}$$

Recall that there is a *cb*-isomorphism $\mathcal{H}_{\mathcal{A}_1} \tilde{\otimes}_{\mathcal{A}_1} \mathcal{A}_2 \approxeq \mathcal{H}_{\mathcal{A}_2}$. Therefore there is a well-defined map

$$\begin{aligned} m^{-1} \colon \mathcal{H}_{\mathcal{A}_2} &\to \mathcal{E}_1 \otimes_{\mathcal{A}_1} \mathcal{A}_2 \\ (a_1, a_2, \dots)^{\intercal} &\mapsto \sum_j x_j^{\alpha} \otimes a_j \end{aligned}$$

We have that the map $m^{-1} \otimes \phi$ inverts m. □

Finally, we need to relate the stuffed modules to the Hilbert C^*-modules.

Proposition 1.2.53 ([28]). *Let $\mathcal{E}$ be stuffed module over an operator pre-C^*-algebra $\mathcal{A}$, Y be a Hilbert C^*-module over an algebra B and $\pi \colon \mathcal{A} \to \operatorname{End}^*_B(Y)$ be a completely contractive algebra homomorphism. Then $\mathcal{E} \tilde{\otimes}_{\mathcal{A}} Y$ is completely isomorphic to a Hilbert C^*-module over B with the inner product given by*

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \lim_{\alpha} \sum_{j=1}^{k_\alpha} \left\langle \langle x_j^{\alpha}, \xi_1 \rangle \eta_1, \langle x_j^{\alpha}, \xi_2 \rangle \eta_2 \right\rangle \tag{1.6}$$

Proof. By definition stuffed modules are direct summands of $\mathcal{H}_{\mathcal{A}}$. We define another operator space structure on $\mathcal{A}$ by

$$\operatorname{Id} \oplus \pi \colon \mathcal{A} \to \mathcal{A} \oplus \operatorname{End}^*_B(Y)$$

This map is completely bounded. Since $\mathcal{H}_{\mathcal{A}}$ remains a rigged module over $\mathcal{A}$ with this structure, by Theorem 1.2.33 we have that $\mathcal{H}_{\mathcal{A}} \tilde{\otimes}_{\mathcal{A}} Y$ is a rigged module over B, and so by Theorem 1.2.32 it is a Hilbert C^*-module. Therefore $\mathcal{E} \tilde{\otimes}_{\mathcal{A}} Y$ is *cb*-isomorphic to a submodule of a Hilbert C^*-module. In the latter C^*-module we have that the formula 1.6 converges since its components constitute an approximate identity for $\mathbb{K}_B(\mathcal{E} \tilde{\otimes}_{\mathcal{A}} Y)$. Thus, by [22][Thm 4.1] this inner product is equivalent to the product we have on $E \tilde{\otimes}_A Y$. □

1.2.11 Operators on Stuffed Modules

In this section, we are going to discuss the operators on stuffed modules. We are going to establish the further similarities between the stuffed modules and Hilbert C^*-modules.

First of all, we would like to study the adjointable operators on stuffed modules. We have the following theorem:

Theorem 1.2.54 (cf. [28]). *Let $\mathcal{A}$ be an operator algebra isomorphic to a pre-C^*-algebra of a C^*-algebra A, and $\mathcal{E}$ be a stuffed right $\mathcal{A}$-module with smooth system given by an approximate unit $\{u_\alpha\}$. If $T, T^* \colon \mathcal{E} \to \mathcal{E}$ are two operators defined on all $\mathcal{E}$ and satisfying $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for all $\xi, \eta \in \mathcal{E}$, then T, T^* are completely bounded and $\mathcal{A}$-linear, i.e. $T, T^* \in \operatorname{CB}^*_{\mathcal{A}}(\mathcal{E})$. Moreover, the cb-norm and operator norm are equivalent to one another and $T \mapsto T^*$ is a well defined cb-anti-isomorphism of $\operatorname{End}^*_{\mathcal{A}}(\mathcal{E})$.*

Proof. The $\mathcal{A}$-linearity of the maps may be checked directly. We would like to prove that T and T^* are completely bounded. To do this, we first dhow that they are bounded. Indeed, we recall that by Lemma 1.2.51 there is a *cb*-anti-isomorphism between $\mathbb{K}_{\mathcal{A}}(\mathcal{E})$ and $\mathcal{E}$, given by $\xi \mapsto \xi^*$. Let M be the maximum of the *cb*-norm of this isomorphism and its inverse.

Now let T, T^* be as in the conditions of the Theorem, and let $\xi \in \mathcal{E}$, $\|\xi\|_{\mathcal{E}} \leq 1$. Set $T_\xi := (T\xi)^* \in \mathbb{K}_{\mathcal{A}}(\mathcal{E})$. Then

$$\|T_\xi(\eta)\|_{\mathcal{E}} = \| \langle T\xi, \eta\rangle \|_{\mathcal{E}} = \| \langle \xi, T^*\eta\rangle \|_{\mathcal{E}} \leq C\|T^*\eta\|_{\mathcal{E}}$$

Thus, using the Banach-Steinhaus theorem, we may conclude that the set

$$\{T_\xi \mid \|\xi\|_{\mathcal{E}} \leq 1\}$$

is bounded, and so $\|T\|_{\mathcal{E}} \leq \infty$. Applying the same considerations to T^*, we obtain that it is bounded as well, and $M^{-1}\|T\|_{\mathcal{E}} \leq \|T^*\|_{\mathcal{E}} \leq M\|T\|_{\mathcal{E}}$. To show that they are completely bounded, observe that

$$\begin{aligned}
{}_m\|(T\xi_{jk})_{jk}\|_{\mathcal{E}} &\leq C^4 \lim_\alpha \lim_\beta \, {}_m\|(\psi_\beta\phi_\beta T\phi_\alpha\psi_\alpha\xi_{jk})_{jk}\|_{\mathcal{E}} \\
&\leq C^4(\sup_{\alpha,\beta} \|\phi_\beta T\psi_\alpha\|_{cb}) \sup_\alpha \, {}_m\|(\psi_\alpha\xi_{jk})_{jk}\|_{\mathcal{E}} \\
&\leq C^5(\sup_{\alpha,\beta} \|\phi_\beta T\psi_\alpha\|) \, {}_m\|(\xi_{jk})_{jk}\|_{\mathcal{E}} \\
&\leq C^7\|T\| \, {}_m\|(\xi_{jk})_{jk}\|_{\mathcal{E}}
\end{aligned}$$

Here C is the common upper bound for ϕ_α and ψ_α, and we also used the fact that $\phi_\beta T\psi_\alpha \colon \mathcal{A}^{k_\alpha} \to \mathcal{A}^{k_\beta}$ is completely bounded since it comes form the multiplication by a matrix with entries in $\mathcal{A}$. We have also shown that $\|T\| \leq \|T\|_{cb} \leq C^7\|T\|$, so that the *cb*-norm is equivalent to the operator norm. □

We emphasize that this result follows precisely the same way as a corresponding result for operators on smooth modules in [28], and here we only make it work in a slightly more general framework of stuffed modules. As we have also seen, the operation $T \mapsto T^*$ is not necessarily completely isometric. However, using Theorem 1.2.16 we may endow $\mathrm{End}^*_{\mathcal{A}}(\mathcal{E})$ with an equivalent operator algebra structure, such that the involution will be completely isometric with respect to this new structure.

Having the notion of adjointable morphisms, we may now prove the following lemma.

Lemma 1.2.55. *Let E be a right stuffed module over an operator pre-C*-subalgebra $\mathcal{A}$ and let $p \in \mathrm{CB}^*(\mathcal{E})$ be a projection on $\mathcal{E}$, i.e. $p^2 = p = p^*$. Then the module $p\mathcal{E}$ is a right stuffed module over A.*

Proof. Since $\mathcal{E}$ *cb*-stabilizable, there are two completely bounded maps $\phi\colon \mathcal{E} \to \mathcal{H}_{\mathcal{A}}$ and $\psi\colon \mathcal{H}_{\mathcal{A}} \to \mathcal{E}$. Define the maps $\phi'_\alpha\colon p\mathcal{E} \to \mathcal{H}_{\mathcal{A}}$ and $\psi'\colon \mathcal{H}_{\mathcal{A}} \to p\mathcal{E}$ by setting $\phi'_\alpha := \phi_\alpha \circ \iota$, where $\iota\colon p\mathcal{E} \to \mathcal{E}$ is the inclusion map, $p \circ \iota = \mathrm{Id}_{p\mathcal{E}}$, and $\psi' := \psi \circ p$. Then, by definition, $p\mathcal{E}$ has a bounded P-quasi unit, and therefore is *cb*-stabilizable. □

Remark 1.2.56. We note that the structure of stuffed module on $p\mathcal{E}$ is then given by an approximate unit

$$v_\alpha = \sum_{i=1}^{k_\alpha} px_j^\alpha \otimes px_j^\alpha$$

Since by the Theorem 1.2.52 there is a completely bounded isomorphism $\mathcal{E}_1 \otimes_{\mathcal{A}_1} \mathcal{A}_2 \approxeq \mathcal{E}_2$ whenever there is a completely bounded inclusion $\mathcal{A}_1 \to \mathcal{A}_2$, we have that any completely bounded adjointable operator T on $\mathcal{E}_1$ extends to a completely bounded adjointable operator on $\mathcal{E}_2$. This observation makes us able to define the notion of regular operators on stuffed modules.

We also define the notion of regular densely defined operators.

Definition 1.2.57. Let $\mathcal{E}$ be a stuffed module over a pre-C^*-algebra $\mathcal{A}$. A densely defined operator $D\colon \mathrm{Dom}D \to \mathcal{E}$ will be called *regular* if

- D^* is densely defined on $\mathcal{E}$,
- $\mathfrak{r}(D) := (1 + D^*D)^{-\frac{1}{2}} \in \mathbb{K}_{\mathcal{A}}(\mathcal{E})$ and $\mathfrak{b}(D) := D\mathfrak{r}(D)$ extends to an operator in $\mathrm{CB}^*_{\mathcal{A}}(\mathcal{E})$.

The notion of selfadjoint regular operator is defined analogously to the case of regular operators on Hilbert C^*-modules.

We shall need the following observation:

Proposition 1.2.58. *In the conditions of 1.2.52, let D be a regular operator on $\mathcal{E}_1$. Then the operator $D \otimes \mathrm{Id}_{\mathcal{A}_2}$ is regular on $\mathcal{E}_2$.*

Proof. The operator $D \otimes \mathrm{Id}_{\mathcal{A}_2}$ is obviously densely defined, so we only need to prove that is resolvent is a compact operator on $\mathcal{E}_2$. But this follows from the fact that

$$\mathfrak{r}(D \otimes \mathrm{Id}_{\mathcal{A}_2}) = (1 + (D^* \otimes \mathrm{Id}_{\mathcal{A}_2})(D \otimes \mathrm{Id}_{\mathcal{A}_2}))^{-\frac{1}{2}} = (1 + D^*D) \otimes \mathrm{Id}_{\mathcal{A}_2} = \mathfrak{r}(D) \otimes \mathrm{Id}_{\mathcal{A}_2}$$

Now, since we have the completely bounded inclusion $\mathcal{E}_1 \hookrightarrow E_2$ and there is a *cb*-isomorphism $\mathcal{E}_i \approxeq \mathcal{E}_i^*$ for $i = 1, 2$, we obtain that the operator $\mathfrak{r}(D) \otimes \mathrm{Id}_{\mathcal{A}_2} \in \mathbb{K}_{\mathcal{A}_2}\mathcal{E}_2$. □

It is still not known for now, whether there could be established a result analogous to the Woronowicz characterization for the case of stuffed modules in general case. However, such a result would be needed in the next chapter, where we generalize the notion of smoothness introduced in [28]. Therefore we are going to give an axiomatic definition of Sobolev chain on stuffed modules.

Definition 1.2.59. Let A be a C^*-algebra $\mathcal{A}$ its operator pre-C^*-subalgebra and $\mathcal{E}$ a stuffed module over $\mathcal{A}$. A regular operator D on $\mathcal{E}$ *induces a Sobolev chain* on $\mathcal{E}$ if we may define a sequence of nested stuffed submodules

$$\cdots \to \mathcal{E}_D^{(j+1)} \to \mathcal{E}_D^{(j)} \to \cdots \to \mathcal{E}_D^{(0)} := \mathcal{E}$$

such that $\mathcal{E}_D^{(j+1)} \approxeq \mathfrak{G}(D_{j+1}) \subseteq \mathcal{E}_D^{(j)} \oplus \mathcal{E}_D^{(j)}$, where $D_0 = D$ and D_{j+1} is a restriction of D_j on $\mathrm{Dom}D_j \approxeq \mathfrak{G}(D_{j+1})$.

This definition may appear to be redundant, because we hypothesize that any regular unbounded operator on a stuffed module generates a Sobolev chain. However, this hypothesis is still unproved, therefore we have to keep the definition.

The proposition we would like to prove concerns the problem of "ring-changing".

Proposition 1.2.60. *Let A be a C^*-algebra $\mathcal{A}$ its operator pre-C^*-subalgebra and $\mathcal{E}$ a stuffed module over $\mathcal{A}$. Suppose that D is an unbounded regular operator on $\mathcal{E}$, inducing a Sobolev chain on $\mathcal{E}$. Let $\mathcal{A}_1$ be another operator pre-C^*-subalgebra of A such that there is a cb-inclusion $\mathcal{A} \hookrightarrow \mathcal{A}_1$. Then there is a dense cb-inclusion $\mathcal{E}^{(j)} \hookrightarrow \mathcal{E}_1^{(j)}$.*

Proof. First of all, observe that $D \otimes \mathrm{Id}_{\mathcal{A}_1}$ is densely defined selfadjoint operator on $\mathcal{E}_1 := \mathcal{E} \tilde{\otimes}_{\mathcal{A}} \mathcal{A}_1$. Consider the operators $\mathfrak{r}(D)$ and $\mathfrak{b}(D)$. We have that $\mathfrak{r}(D) \otimes \mathrm{Id}_{\mathcal{A}_1}$ and $\mathfrak{b}(D) \otimes \mathrm{Id}_{\mathcal{A}_1}$ are well defined compact resp. completely bounded operators on $\mathcal{E}_1$. Therefore the Woronowicz projection $p_{(D\otimes \mathrm{Id}_{\mathcal{A}_1})}$ is a well-defined completely bounded operator on $\mathcal{E}_1 \oplus \mathcal{E}_1$. Moreover, by its construction we have that $p_{D\otimes \mathrm{Id}_{\mathcal{A}_1}} = p_D \otimes \mathrm{Id}_{\mathcal{A}_1}$. But, by definition

$$\mathcal{E}_{1,D}^{(1)} \approxeq p_{(D\otimes \mathrm{Id}_{\mathcal{A}_1})}\mathcal{E}_1^{\oplus 2} \approxeq (p_D \otimes \mathrm{Id}_{\mathcal{A}_1})(\mathcal{E}_D^{(1)} \tilde{\otimes}_{\mathcal{A}} \mathcal{A}_1)^{\oplus 2} \approxeq (p_D \mathcal{E}_D^{(1)})^{\oplus 2} \tilde{\otimes} \mathcal{A}_1$$

Thus $\mathcal{E}_1^{(1)} \approxeq \mathcal{E}^{(1)} \tilde{\otimes}_{\mathcal{A}} \mathcal{A}_1$. Since $\mathcal{E}^{(1)}$ was *cb*-stabilizable, so is $\mathcal{E}_1^{(1)}$.

Applying the same reasoning to $\mathcal{E}^{(1)}$ we obtain the that $\mathcal{E}_{1,D}^{(2)} = \mathcal{E}_D^{(2)} \tilde{\otimes}_{\mathcal{A}} \mathcal{A}_1$, and so on. Since $\mathcal{A} \hookrightarrow \mathcal{A}_1$ was a *cb*-inclusion, so will by 1.2.52 be the map $\mathcal{E}_D^{(n)} \to \mathcal{E}_{1,D}^{(n)}$. □

1.2.12 Connections

In this subsection we assume all the operator algebras to be unital.

Connections is an important geometrical notion, that is carried to noncommutative geometry and unbounded KK-theory. Connections are inevitable part of the construction of the product of unbounded operators. Therefore we have to consider them here in detail.

The first step in the construction of the connections is the definition of 1-forms.

Definition 1.2.61. Let $\mathcal{A}$ be an operator algebra. The module of *universal* 1*-forms* on $\mathcal{A}$ is defined as

$$\Omega^1(\mathcal{A}) := \ker(m\colon \mathcal{A} \tilde{\otimes} \mathcal{A} \to \mathcal{A})$$

By this definition, there is an exact sequence of operator modules

$$0 \to \Omega^1(\mathcal{A}) \to \mathcal{A} \tilde{\otimes} \mathcal{A} \xrightarrow{m} \mathcal{A} \to 0$$

and $\Omega^1(\mathcal{A})$ inherits a grading form $\mathcal{A}$ whenever $\mathcal{A}$ is graded.

There is a natural graded derivation on $\mathcal{A}$ given by the map

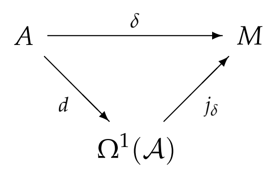
$$\begin{aligned} d\colon \mathcal{A} &\to \Omega^1(\mathcal{A}) \\ a &\to 1 \otimes a - (-1)^{\partial a} a \otimes 1 \end{aligned}$$

and one may observe that every element of $\Omega^1(\mathcal{A})$ has a form adb. The $\mathcal{A}$-bimodule structure on $\Omega^1(\mathcal{A})$ is then given by $(adb)\cdot c = ad(bc) + (-1)^{\partial b} abdc$.

The involution on $\mathcal{A}$ induces a natural involution on $\Omega^1(\mathcal{A})$, defined by

$$(adb)^* := -(-1)^{\partial b}(db^*)\cdot a^*$$

Lemma 1.2.62 ([28])**.** *The derivation d is universal in the sense that for any completely bounded graded derivation $\delta\colon \mathcal{A}\to M$ into an operator $\mathcal{A}$ bimodule there is a unique completely bounded bimodule homomorphism $j_\delta\colon \Omega^1(\mathcal{A})\to M$ such that the diagram*



is commutative. If δ is homogeneous, then so is j_δ and $\partial\delta = \partial j_\delta$.

Any derivation $\delta\colon \mathcal{A}\to M$ has its associated module of 1-forms

$$\Omega^1_\delta := j_\delta(\Omega^1(\mathcal{A})) \subseteq M$$

Recall that for any stuffed $\mathcal{A}$-module $\mathcal{E}$ there is a well defined $\mathcal{A}$-valued inner product on $\mathcal{E}$. This inner product induces a pairing

$$\begin{array}{rcl} \mathcal{E}\times\mathcal{E}\,\tilde{\otimes}_{\mathcal{A}}\,\Omega^1(\mathcal{A}) & \to & \Omega^1(\mathcal{A}) \\ \langle \xi_1, \xi_2\otimes\omega\rangle & \to & \langle\xi_1,\xi_2\rangle\omega \end{array}$$

We shall abusively write $\langle \xi_1, \xi_2\otimes\omega\rangle$ for this pairing. We may also define a pairing

$$\mathcal{E}\,\tilde{\otimes}_{\mathcal{A}}\,\Omega^1(\mathcal{A})\times\mathcal{E}\to\Omega^1(\mathcal{A})$$

by setting $\langle\xi_1\otimes\omega,\xi_2\rangle := \langle\xi_1,\xi_2\otimes\omega\rangle^*$.

Definition 1.2.63 ([28])**.** Let $\delta\colon\mathcal{A}\to M$ be a (graded) derivation as above, and let E be a right operator $\mathcal{A}$-module. A *δ-connection* on E is a completely bounded (even) linear map

$$\nabla_\delta\colon \mathcal{E}\to\mathcal{E}\,\tilde{\otimes}_{\mathcal{A}}\,\Omega^1_\delta(\mathcal{A})$$

satisfying the Leibnitz rule

$$\nabla_\delta(\xi a) = \nabla(\xi)\cdot a + \xi\otimes\delta(b)$$

If $\delta = d$, the connection will be denoted just as ∇ and is referred to as a *universal connection*. In case when E is stuffed module over an operator pre-C^*-algebra, then the connection ∇ will be called $*$*-connection* if there is a connection ∇^* on $\mathcal{E}$ for which

$$\langle\xi_1,\nabla\xi_2\rangle - \langle\nabla^*\xi_1,\xi_2\rangle = (-1)^{\partial\langle\xi_1,\xi_2\rangle} d\langle\xi_1,\xi_2\rangle$$

If $\nabla = \nabla^*$ the connection will be called *Hermitian*.

We note that by the universality property mentioned above any universal connection ∇ on a stuffed module induces a δ-connection E for any completely bounded derivation δ. This is done by setting $\nabla_\delta := \mathrm{Id}_{\mathcal{E}} \otimes j_\delta \circ \nabla$. As in [28], we adopt the notation ∇_S for the connection induced by the derivation $\delta(\cdot) = [S, \cdot]$ for $S \in \mathrm{CB}_{\mathbb{C}}(X, Y)$, where X and Y are operator $\mathcal{A}$-modules.

The existence of a universal connection on a given module is a strong condition. It was shown by Cuntz and Quillen in [18] that the universal connections characterize algebraic projectivity of the module.

Theorem 1.2.64 ([18, 28])**.** *A right operator $\mathcal{A}$-module $\mathcal{E}$ admits a universal connection if and only if the multiplication map $m\colon \mathcal{E} \tilde{\otimes} \mathcal{A} \to \mathcal{E}$ is $\mathcal{A}$-split.*

Corollary 1.2.65 ([28])**.** *Let $\mathcal{A}$ be an operator pre-C^*-algebra and $\mathcal{E}$ be a stuffed module over $\mathcal{A}$. Then $\mathcal{E}$ admits a Hermitian connection.*

Proof. Since $\mathcal{E}$ is a stuffed $\mathcal{A}$-module, it has a stabilization property, i.e. $\mathcal{E} \oplus \mathcal{H}_{\mathcal{A}} \approxeq \mathcal{H}_{\mathcal{A}}$. Therefore there is an operator $p \in \mathrm{CB}^*_{\mathcal{A}}(\mathcal{E})$, $p = p^2 = p^*$, such that $\mathcal{E} = p\mathcal{H}_{\mathcal{A}}$. Observing that $\mathcal{H}_{\mathcal{A}} \tilde{\otimes}_{\mathcal{A}} \Omega^1(\mathcal{A}) \approxeq \mathcal{H} \tilde{\otimes} \Omega^1(\mathcal{A})$, we may construct a *Grassmannian connection*

$$\begin{aligned} d\colon \mathcal{H}_{\mathcal{A}} &\to \mathcal{H} \tilde{\otimes} \Omega^1(\mathcal{A}) \\ h \otimes a &\mapsto h \otimes da \end{aligned}$$

which is Hermitian. Since p is a projector, so the connection $p\nabla p\colon E \to E \tilde{\otimes}_{\mathcal{A}} \Omega^1(\mathcal{A})$ will also be Hermitean. □

It is easy to see that the connections are forming an affine space. More precisely, if ∇ and ∇' are two universal connections on an operator $\mathcal{A}$-module $\mathcal{E}$, then $(\nabla - \nabla')\colon \mathcal{E} \to \Omega^1(\mathcal{A})$ is a completely bounded $\mathcal{A}$-linear operator. Indeed, the complete boundedness follows from definition, and as to $\mathcal{A}$-linearity we have that

$$(\nabla - \nabla')(\xi a) = (\nabla \xi)a + \xi \otimes da - (\nabla' \xi)a - \xi \otimes da = (\nabla - \nabla')(\xi)a$$

This observation allows us to prove the following

Lemma 1.2.66. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two operator pre C^*-subalgebras of a C^*-algebra A, E be a Hilbert C^*-module over A and let an approximate unit $\{u_\alpha\}$ define a structure of stuffed module on E for both $\mathcal{A}_1$ and $\mathcal{A}_2$. If the inclusion $\mathcal{A}_1 \to \mathcal{A}_2$ is completely bounded, then for every connection $\nabla_1\colon \mathcal{E}_1 \to \mathcal{E}_1 \tilde{\otimes}_{\mathcal{A}_1} \Omega^1(\mathcal{A}_1)$ there is a canonically associated connection $\nabla_2\colon \mathcal{E}_2 \to \mathcal{E}_2 \tilde{\otimes}_{\mathcal{A}_2} \Omega^1(\mathcal{A}_2)$. Moreover, if ∇_1 is Hermitian, then so is ∇_2.*

Proof. Using the identification $\mathcal{E}_2 \approxeq \mathcal{E}_1 \tilde{\otimes}_{\mathcal{A}_1} \mathcal{A}_2$ form the Proposition 1.2.52, we observe that there is a canonical *cb*-isomorphism

$$\begin{aligned} \mathcal{A}_2 \tilde{\otimes} \Omega^1(\mathcal{A}_1) \tilde{\otimes} \mathcal{A}_2 &\to \Omega^1(\mathcal{A}_2) && (1.7) \\ a_1 \otimes b_1 db_2 \otimes a_2 &\mapsto (a_1 b_1 db_2 a_2 - a_1 b_1 b_2 da_2) && (1.8) \end{aligned}$$

which is compatible with d. This allows us to define

$$\nabla_2(\xi \otimes a) := \nabla_1(\xi) \otimes a + \xi \otimes 1_{\mathcal{A}_2} da$$

The uniqueness of ∇_2 follows from the fact that $\mathcal{E}_1$ is dense in $\mathcal{E}_2$. The fact that ∇_2 is Hermitian when ∇_1 is Hermitian may be then shown by the direct calculation. □

Chapter 2

Unbounded KK-Theory

2.1 Smooth Systems on C^*-Algebras

We have already briefly discussed different approaches to the definition of the smooth systems on C^*-algebras in the Introduction. Now it is time to introduce the notion of the smooth system that we shall use throughout the rest of the paper. In the subsequent subsections we shall first give the definition of smooth system on a C^*-algebra, and then establish its connection to the unbounded KK-theory. We shall also present examples, showing how the notion of smoothness coming out o the unbounded KK-theory may substantively deviate from the one which is habitual in differential and noncommutative geometry.

2.1.1 Smooth Systems, First Fréchetization and $\Psi^{(\bullet)}$ Sets

We start with the definition of the smooth systems on C^*-algebras.

Definition 2.1.1. Let A be a C^*-algebra. An *n-smooth system* (or C^n-system) $\mathscr{A}$ on A is an inverse system of pre-C^*-subalgebras of A

$$(\cdots \subseteq)\mathcal{A}^{(n)} \subseteq \mathcal{A}^{(n-1)} \subseteq \cdots \subseteq \mathcal{A}^{(1)} \subseteq \mathcal{A}^{(0)} := A$$

such that all $\mathcal{A}^{(j)}$, $j = 0, \ldots, n$ are isomorphic to operator algebras with completely isometric involution induced by the invoultion on A. These operator algebras will be abusively denoted by $\mathcal{A}^{(j)}$, and we demand that the operator maps $\mathcal{A}^{(k)} \hookrightarrow \mathcal{A}^{(k-1)}$ are completely bounded, essential and involutive for $k = 1, \ldots, n$. The smooth system will be called ∞-*smooth* (or C^∞-system) if the system of these subalgebras is infinite and the inverse limit of the system $\mathcal{A}^{(\infty)}$ is also a pre-C^*-algebra. The number n (including the case $n = \infty$) will be called the *order of smoothness* of the smooth system and will be denoted by $\mathrm{ord}(\mathscr{A})$.

We shall denote by ${}_m\|\cdot\|_n$ the operator norms on $\mathcal{A}^{(n)}$ and for simplicity also demand that $\|1\|_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ in case when the algebras are unital.

We shall also refer to the smooth system with $\mathcal{A}^{(n)} = A$ for all n as *trivial smooth system*.

Two smooth systems $\mathscr{A}_1$ and $\mathscr{A}_2$ will be called *equivalent* if there is a *cb*-isomorphism $\mathcal{A}_1^{(n)} \approxeq \mathcal{A}_2^{(n)}$ for all $n \leq \operatorname{ord}(\mathscr{A}_1) = \operatorname{ord}(\mathscr{A}_2)$.

Though in the Definition 2.1.1 we use operator algebras instead of Banach algebras, this approach is in some sense even more general then the one in [6]. We are now going to develop the framework that will relate it to the unbounded KK-theory.

Definition 2.1.2. For given two C^*-algebras A and B, an unbounded (A, B)-KK-cycle (E, D) and a natural number k the *first frécetization* is a map $\mu\colon (A, B, E, D, k) \to \mathcal{A}_{\mu,D}^{(k)}$, where $\mathcal{A}_{\mu,D}^{(k)}$ is an operator algebra isomorphic to a subalgebra of A.

Definition 2.1.3. A *smooth system on an algebra A generated by operator D with respect to the fréchetization μ* is the longest sequence of nested subalgebras $\mathcal{A}_{\mu,D}^{(k)}$ of A with the starting point $\mathcal{A}^{(0)} = A$ satisfying the conditions of smooth system. We shall denote this system by $\mathscr{A}_{\mu,D}$. Following [28], in case when $\mathcal{A} = \operatorname{End}_B(E)$ we shall denote the smooth system $(\operatorname{End}_B(E))_{\mu,D}^{(n)} =: \operatorname{Sob}_{\mu,D}^{(n)}$

Definition 2.1.4. Let $\mathscr{A}$ be a smooth system on a C^*-algebra A with $\operatorname{ord}\mathscr{A} \geq n$, $n \in \mathbb{N}$, and let B be another C^*-algebra. We shall say that the unbounded (A, B)-KK-cycle (E, D) is *n-smooth (C^n) with respect to $\mathscr{A}$* if $\mathcal{A}^{(k)} \hookrightarrow \mathcal{A}_{\mu,D}^{(k)}$ for all $k \leq n$, and the inclusion morphism induces a completely bounded homomorphism of operator algebras. The set of all such cycles will be denoted by $\Psi_\mu^{(n)}(\mathscr{A}, B)$. We say that $(E, D) \in \Psi_\mu^{(\infty)}(\mathscr{A}, B)$ if $\operatorname{ord}(\mathscr{A}) = \infty$ and $(E, D) \in \Psi_\mu^{(n)}(\mathscr{A}, B)$ for all $n \in \mathbb{N}$. Note that in this case $\mathcal{A}_{\mu,D}^{(\infty)}$ will automatically be a pre-C^*-algebra.

We may immediately observe that if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two smooth systems on a C^*-algebra, A, such that $\mathcal{A}_1^{(k)} \subseteq \mathcal{A}_2^{(k)}$ and the induced map of operator pseudoalgebras completely bounded for all $k \leq n$, then by definition we have that $\Psi_\mu^{(n)}(\mathscr{A}_1, B) \subseteq \Psi_\mu^{(n)}(\mathscr{A}_2, B)$ for any C^*-algebra B. If the smooth systems are equivalent, then $\Psi_\mu^{(n)}(\mathscr{A}_1, B) = \Psi_\mu^{(n)}(\mathscr{A}_2, B)$.

Remark 2.1.5. It is important to note that in fact the notions of smooth systems and fréchetizations may be defined in a more general way than we have described. Namely, the operaotor algebras with involutions may be replaced with operator pseudoalgebras with pseudo-involutions. However, the results 1.2.8 and 1.2.16 show that we may always reduce the task to the case of operator algebras with involutions. Therefore we are going to stick to the concrete algebra approach, since it simplifies the explanations, and put the remarks to indicate, how would it be possible to generalize the results to the case of pseudoalgebras with pseudoinvolutions.

2.1.2 Relation to Classical KK-Theory

All the theory we have been developed before, including the previous section, appears to be very abstract. In this subsection we are finally going to establish the relation between the definitions we have given in the previous one to the KK-theory, and several of the next ones will be devoted to the examples provided by this result. Before formulating it, we need to outline the type of fréchetizations that naturally arise in the topics related to the unbounded KK-theory.

We shall use the notation $\mathrm{ad}_b(a) := ba - (-1)^{\partial a \partial b} ab$ for the graded commutator operation, and $\mathrm{ad}_b^n(a)$ for its n'th power.

Definition 2.1.6. Let μ be a fréchetization. We shall call μ *commutator bounded* if for all $n \in \mathbb{N}$ there exists a positive number C_n such that

$$\|a\|_{\mathcal{A}_D^{(n)}} \leq C_n \max\{\|a\|, \|\mathrm{ad}_D(a)\|, \|\mathrm{ad}_D^2(a)\|, \dots, \|\mathrm{ad}_D^n(a)\|\}$$

Here we include the case when $\mathrm{ad}_D^k(\pi(a))$ does not extend to a bounded operator on E, setting $\|\mathrm{ad}_D^k(a)\| = \infty$.

The fréchetization will be called *analytic* if for any smooth system $\mathscr{A}_{\mu,D}$ we have that the norm $\|\cdot\|_{\mu,n,D}$ is analytic with respect to $\|\cdot\|_{\mu,n-1,D}$ for all $1 \leq n \leq \mathrm{ord}(\mathscr{A}_{\mu,D})$.

The fréchetization will be called *differential* if for any smooth system $\mathscr{A}_{\mu,D}$ with $\mathrm{ord}(\mathscr{A}_{\mu,D}) = \infty$ the ordered set $(\|\cdot\|, \|\cdot\|_{1,\mu,D}, \|\cdot\|_{2,\mu,D}, \dots)$ is a differential seminorm on $\mathscr{A}_{\mu,D}$.

Now we are ready to formulate the main result of the subsection.

Theorem 2.1.7. *Let μ be a fréchetization, which is commutator bounded. Suppose also that it is either analytic or differential (or both). Then, for any separable unital C^*-algebra A and any set of isomorphism classes of C^*-alberas Λ there is an ∞-smooth system $\mathscr{A}$ on A that for any $B \in \Lambda$ there is a surjective map $\Psi_\mu^{(\infty)}(\mathscr{A}, B) \to \mathrm{KK}(A, B)$, induced by the bounded transform map.*

Before we proceed to the proof, we shall discuss some more specific formulations, allowing us to apply it in concrete situations. First of all it should be noted that the notion of first fréchetization was introduced by the author because he has encountered different ways to define the C^n-smooth algebra by means of an unbounded KK-cycle. Further in the text we shall stick to the fréchetization that is given by the definition of smooth algebra introduced by Mesland in [28], which satisfy the conditions of the theorem with the constant $C_n = 2^n$. This particular construction has been widely studied in [28] and will be playing a crucial role in our further development of the unbounded KK-theory.

As to the set Λ mentioned in the formulation, we may have the following examples.

Example 2.1.8. Let $\Lambda = \{\mathbb{C}\}$. Then the unbounded KK-cycles in $\Psi_\mu^{(n)}(\mathscr{A}, B)$ are unbounded K-cycles, that include spectral triples. It should be noted, however, as we shall see from examples, the smooth system $\mathscr{A}$ may be very far from, for instance, the ones coming from differential geometry, even when the algebra A was an algebra of smooth functions on a smooth manifold.

Example 2.1.9. It may be shown that the separable metric spaces up to an isomorphism form a set. Therefore, the same is true for isomorphism classes of separable C^*-algebras. Thus for any separable C^*-algebra A we may choose a unique smooth system $\mathscr{A}$, that for any separable C^*-algebra B there will be a surjective map $\Psi_\mu^{(n)}(\mathscr{A}, B) \to \mathrm{KK}(A, B)$

In order proceed to the proof of the Theorem 2.1.7 we first need to prove following lemmas.

Lemma 2.1.10. *Let A be a separable C^*-algebra. Then for any C^*-algebra B and any element $[(E,F)] \in \mathrm{KK}(A,B)$ there exists an unbounded (A,B)-KK-cycle (E,D), such that $[(E,\mathfrak{b}(D))] = [(E,F)]$ and the set of such $a \in A$, that $\mathrm{ad}^n_D(a)$ extends to a bounded operator on E, is dense in A.*

Proof. This result is a generalization of the Theorem 17.11.4 form [5]. Fix a total system $\{a_j\}$ of A. For given F there exists a strictly positive element $h \in \mathbb{K}(E)$ of degree 0 which commutes with F [5]. Now, according to [30, 3.12.14] there exists an approximate unit u_k for $\mathbb{K}(E)$, contained in $C^*(h)$, quasicentral for A, with the property that $u_k \geq 0$, $u_{k+1} \geq u_k$ and $u_{k+1}u_k = u_k$ for all $k \in \mathbb{N}$. Denote $d_k = u_{k+1} - u_k$. Passing, when needed, to a subsequence, we may assume that $\|d_k[F;a_j]\| < 2^{-k^2}$ and $\|[d_k;a_j]\| < 2^{-k^2}$ for all $k \geq j+1$. Set $X = \widehat{C^*(h)} \approx \sigma(h) \setminus \{0\}$ and let X_n be the support of u_k. Then $\langle X_k \rangle$ is an increasing sequence of compact subsets of X and $X = \bigcup_{k=1}^{\infty} X_k$. Put

$$r_k = \sum_{l=1}^{k} 2^l d_l$$

This sequence converges pointwise on X to an unbounded function r. Observe that $r \geq 2^k$ on $X \setminus X_k$, so that $R = r^{-1}$ defines an element of $C^*(h)$. Note also, that d_k defines a bounded function on the space X and, since $\|d_k\| \leq 1$ and $d_k d_{k-l} = 0$ for all $k \geq 3$ and $2 \leq l \leq k-1$, we obtain that

$$\|r_k\| \leq \max_{l=2,\dots,k}\{\|2^{l-1}d_{l-1} + 2^l d_l\|\} \leq 3 \cdot 2^{k-1} < 2^{k+1}$$

Let now $D = Fr$. Then $D = D^*$ and $(1+D^2)^{-1}$ extends to $R^2(1+R^2)^{-1} \in \mathbb{K}_A(E)$. We would like to prove that $\mathrm{ad}^n_D(a_j)$ extends to a bounded operator on E for all $n \in \mathbb{N}$. To do this we first observe that, since F, r_k and d_k commute for all k,

$$\begin{aligned}
\mathrm{ad}^n_{Fr_{k+1}}(a_j) - \mathrm{ad}^n_{Fr_k}(a_j) =& \mathrm{ad}^n_{F(r_k + 2^{k+1}d_{k+1})}(a_j) - \mathrm{ad}^n_{Fr_k}(a_j)\\
&= \sum_{l=0}^{n} C^l_n \mathrm{ad}^{n-l}_{Fr_k}\left(\mathrm{ad}^l_{F\cdot 2^{k+1}d_{k+1}}(a_j)\right) - \mathrm{ad}^n_{Fr_k}(a_j)\\
&= \sum_{l=1}^{n} 2^{l(k+1)} C^l_n \mathrm{ad}^{n-l}_{Fr_k}\left(\mathrm{ad}^l_{Fd_{k+1}}(a_j)\right)\\
&= \sum_{l=1}^{n} 2^{l(k+1)} C^l_n \mathrm{ad}^{n-l}_{Fr_k}\left(\mathrm{ad}^{l-1}_{Fd_{k+1}}([Fd_{k+1};a_j])\right)\\
&= \sum_{l=1}^{n} 2^{l(k+1)} C^l_n \mathrm{ad}^{n-l}_{Fr_k}\left(\mathrm{ad}^{l-1}_{Fd_{k+1}}(F[d_{k+1};a_j] + [F;a_j]d_{k+1})\right)
\end{aligned}$$

where C^l_n are binomial coefficients. Now since $\|F\| = 1$, $\|d_{k+1}\| \leq 1$ and $\|r_k\| < 2^{k+1}$, we obtain that $\|\mathrm{ad}_{Fd_{k+1}}(b)\| \leq 2\|b\|$ and $\|\mathrm{ad}_{Fr_k}(b)\| \leq 2^{k+2}\|b\|$ for any bounded operator b, we estimate for $k \geq j+1$:

$$
\begin{aligned}
&\left\| \sum_{l=1}^{n} 2^{l(k+1)} C_n^l \mathrm{ad}_{Fr_k}^{n-l} \left(\mathrm{ad}_{Fd_{k+1}}^{l-1} (F[d_{k+1}; a_j] + [F; a_j] d_{k+1}) \right) \right\| \\
\leq & \sum_{l=1}^{n} 2^{l(k+1)} C_n^l \left\| \mathrm{ad}_{Fr_k}^{n-l} \left(\mathrm{ad}_{Fd_{k+1}}^{l-1} (F[d_{k+1}; a_j] + [F; a_j] d_{k+1}) \right) \right\| \\
= & \sum_{l=1}^{n} 2^{l(k+1)} C_n^l \cdot 2^{(k+2)(n-l)} \left\| \mathrm{ad}_{Fd_{k+1}}^{l-1} (F[d_{k+1}; a_j] + [F; a_j] d_{k+1}) \right\| \\
\leq & \sum_{l=1}^{n} 2^{l(k+1)} C_n^l \cdot 2^{(k+2)(n-l)} \cdot 2^{l-1} \left\| F[d_{k+1}; a_j] + [F; a_j] d_{k+1} \right\| \\
\leq & \sum_{l=1}^{n} 2^{l(k+1)} C_n^l \cdot 2^{(k+2)(n-l)} \cdot 2^{l-1} \cdot (2^{-k^2} + 2^{-k^2}) \\
= & \sum_{l=1}^{n} C_n^l 2^{l(k+1)+(k+2)(n-l)+(l-1)+1-k^2} \\
= & \sum_{l=1}^{n} C_n^l 2^{kn+2n-k^2} \\
= & 2^n \cdot 2^{kn+2n-k^2} \\
= & 2^{-k^2+(k+3)n}
\end{aligned}
$$

Summing up these estimates, we obtain that

$$\|\mathrm{ad}_{Fr_{k+1}}^n(a_j) - \mathrm{ad}_{Fr_k}^n(a_j)\| \leq 2^{-k^2+(k+3)n}$$

and so the sequence $\{\mathrm{ad}_{Fr_k}^n(a_j)\}$ is norm convergent in $k \to \infty$.

Now, by the construction, we may write

$$\mathrm{ad}_D^n(a_j)\xi = \lim_{k\to\infty} \mathrm{ad}_{Fr_k}(a_j)\xi$$

when the limit exists. We have just proved that the this limit exists for all $\xi \in E$. Therefore $\mathrm{ad}_D(a_j)$ is defined on a dense subspace of E and coincides (in this subspace) with a bounded operator $\lim_{k\to\infty} \mathrm{ad}_{Fr_k}(a_j)$. Hence the operator $\mathrm{ad}_D^n(a_j)$ extends to a bounded operator on E. Thus, since $\{a_j\}$ form a total set for A, the set of all $a \in \mathcal{A}$ such that $\mathrm{ad}_D^n(a)$ extends to a bounded operator on E is dense in $\mathcal{A}$. Pointing out, that it is true for $n = 1$ and observing that

$$D(1+D^2)^{-1/2} = F(1+R^2)^{-1/2}$$

and the latter operator is a "compact perturbation" of F, we obtain that (E, D) is an unbounded (A, B)-KK-cycle and that $[(E, F)] = [(E, \mathfrak{b}(D))]$. QED. □

We have actually shown more than we have claimed in the formulation of Lemma 2.1.10. Namely, we proved that for any element (E, F) and any total system $\{a_j\}$ one we may construct such D that

$$\|\mathrm{ad}_D^n(a_j)\| \leq c_{n,j} \tag{2.1}$$

where $c_{n,j}$ is a positive number that does not depend neither on the choice of F nor on $\{a_j\}$. This observation lets us prove the next lemma.

Lemma 2.1.11. *Let μ be some fréchetization, A be a separable C^*-algebra, $\{a_j\}$ - an arbitrary total system on A and Ω be a set of such unbounded (A, B_ω)-KK-cycles (E_ω, D_ω) that*

- $\operatorname{ord}(\mathscr{A}_{\mu,D_\omega}) = \infty$
- *For each n and the operator algebra $\mathcal{A}^{(n)}_{\mu,n,D_\omega}$ we have that $\|a_j\|_{\mu,n,D_\omega} \leq K_{n,j}$, where $K_{n,j}$ are some positive numbers independent of $(E_\omega, D_\omega) \in \Omega$.*

Then there is an infinite nested system of dense subalgebras $\mathcal{A}^{(n)}$ in A, satisfying all the properties of smooth system except, possibly, for holomorphic stability and essentiality of the maps $\mathcal{A}^{(n)} \hookrightarrow \mathcal{A}^{(n-1)}$, such that $\mathcal{A}^{(n)} \subseteq \mathcal{A}^{(n)}_{\mu,D}$ and the map induced by this inclusion is completely bounded.

Proof. We iteratively define matrix norms

$$_m\|(a_{ik})\|'_n := \max\{\sup_{\omega\in\Omega}\ _m\|(a_{ik})\|_{\mu,n,D};\ _m\|(a_{ik})\|'_{n-1}\}$$

with $\|a\|_0$ being the C^*-norm on A and set $\mathcal{A}^{(n)}_1$ to be the completion of $\operatorname{span}(\{a_j\})$ in the norm $\|\cdot\|'_n$. By the construction $\|a_j\|'_n \leq K_{j,n}$ and so $\mathcal{A}^{(n)}_1$ are dense in A. It is also obvious that the sets $\mathcal{A}^{(n)}_1$ are actually subalgebras of A (one may use the triangle inequality to check this).

The matrix norms $_m\|\cdot\|'_n$ are finite for all $(a_{ik})_{ik} \in M_n(\mathcal{A}^{(n)}_1)$ since we may estimate

$$_m\|(a_{ik})\|_{\mu,n,D} \leq m^2 \max_{1\leq i,k\leq m}\ _1\|a_{ik}\|_{\mu,n,D}$$

One then may check directly that the collection of matrix norms $\{\ _m\|\cdot\|_{\mu,n,D}\}_{m=1}^\infty$ makes $\mathcal{A}^{(n)}_1$ into an $\mathcal{L}^\infty$ matricially normed space. It is also easy to check that the multiplication on $\mathcal{A}^{(n)}_1$ is completely contractive. Indeed, $\mathcal{A}^{(0)}_1 = A$, so the claim holds for $n = 0$. Suppose that it is true for $n-1$. Then for n we have

$$\begin{aligned}
&\|(a)_{ik}(b)_{pq}\|'_n \\
&\leq \max\{\|(a)_{ik}(b)_{pq}\|'_{n-1};\ \sup_{\omega\in\Omega}\|(a)_{ik}(b)_{pq}\|_{\mu,n,D_\omega}\} \\
&\leq \max\{\|(a)_{ik}\|'_{n-1}\ _m\|(b)_{pq}\|'_{n-1};\ \sup_{\omega\in\Omega}\|(a)_{ik}\|_{\mu,n,D_\omega}\sup_{\omega\in\Omega}\|(b)_{pq}\|_{\mu,n,D_\omega}\} \\
&\leq \max\{\|(a)_{ik}\|'_{n-1};\ \sup_{\omega\in\Omega}\|(a)_{ik}\|_{\mu,n,D_\omega}\}\max\{\|(b)_{pq}\|'_{n-1};\ \sup_{\omega\in\Omega}\|(b)_{pq}\|_{\mu,n,D_\omega}\} \\
&= \|(a)_{ik}\|'_n\|(b)_{pq}\|'_n
\end{aligned}$$

Observe also that by definition the involution on $\mathcal{A}^{(n)}_{\mu,D}$ is a completely isometric anti-isomorphism. Therefore, the involution of A induces a completely isometric involution on

$\mathcal{A}_1^{(n)}$ as an operator space. Thus, by Theorem 1.2.16 $\mathcal{A}_1^{(n)}$ is *cb*-isomorphic to an involutive operator algebra. We denote this operator algebra by $\mathcal{A}^{(n)}$.

By the construction we also have that $\mathcal{A}_1^{(n)} \subseteq \mathcal{A}_1^{(n-1)}$ in the sense of subalgebras of A, and that

$$\|(a_{ik})\|'_{\mu,n,D} \leq \|(a_{ik})\|'_n$$

Hence, the inclusion map $\mathcal{A}_1^{(n)} \to \mathcal{A}_{\mu,D}^{(n)}$ is completely contractive. Therefore the map $\mathcal{A}^{(n)} \to \mathcal{A}_{\mu,D}^{(n)}$ induced by the same inclusion is indeed completely bounded. □

As in definition, we shall use the notation ${}_m\|\cdot\|_n$ for the operator norms on the operator algebra $\mathcal{A}^{(n)}$ constructed in the Lemma 2.1.11.

Now we are finally may impose the conditions that will guarantee us that the nested system of algebras $\{\mathcal{A}^{(n)}\}$ is a smooth system.

Lemma 2.1.12. *Let μ be a commutator bounded analytic fréchetization. Then for any unital separable C^*-algebra A and any set $\{(E_\omega, F_\omega)\}_{\omega\in\Omega}$ of* KK*-cycles over (A, B_ω) there exists an ∞-smooth system $\mathscr{A}$ such that the map $\Psi_\mu^{(\infty)}(A, B_\omega) \to \{[(E_\omega, F_\omega)]\}_{\omega\in\Omega}$ is surjective.*

Proof. Without loss of generality we may suppose $F^2 = 1$ and $F = F^*$. Choose a total system $\{a_j\}$ on A and construct the unbounded KK-cycles (E_ω, D_ω) for each (E_ω, F_ω) by the method described in Lemma 2.1.10. Since μ is commutator bounded, we have that

$$\|a_j\|_{\mu,n,D_\omega} \leq C_n \max\{\|a\|, \|\mathrm{ad}_{D_\omega}(a)\|, \ldots, \|\mathrm{ad}^n_{D_\omega}(a)\|\} \leq C_n \max_{k=0,\ldots,n}(c_{k,j}) =: K_{n,j} \tag{2.2}$$

and $K_{n,j}$ does not depend on ω. Thus we may apply the Lemma 2.1.11. We denote the resulting sequence of algebras $\mathscr{A} = \{\mathcal{A}^{(n)}\}$. To prove that $\mathscr{A}$ is a smooth system, we only need to prove the holomorphic stability of the algebras $\mathcal{A}^{(n)}$.

But since μ is analytic we have that $\|\cdot\|_{\mu,n,D_\omega}$ is analytic with respect to $\|\cdot\|_{\mu,n-1,D_\omega}$ for all n. Since the fréchetization μ is analytic, we have that for all $a \in A$ such that $\sup_{\omega\in\Omega}\|a\|_{\mu,n-1,D_\omega} \leq 1$ we have that

$$\limsup_{m\to\infty} \frac{\ln(\sup_{\omega\in\Omega}\|a^m\|_{\mu,n,D_\omega})}{m} = \limsup_{m\to\infty} \frac{\sup_{\omega\in\Omega}\ln\|a^m\|_{\mu,n,D_\omega}}{m} \leq 0$$

Therefore the Banach norm $\sup_{\omega\in\Omega}\|\cdot\|_{\mu,n-1,D_\omega}$ is analytic with respect to $\sup_{\omega\in\Omega}\|\cdot\|_{\mu,n,D_\omega}$. Since the norm $\|\cdot\|_n$ on $\mathcal{A}^{(n)}$ is equivalent to the norm $\sup_{\omega\in\Omega}\|a\|_{\mu,n,D_\omega}$ for all n, we have that $\mathcal{A}^{(n)}$ are stable under holomorphic functional calculus on A. The holomorphic stability of $\mathcal{A}^{(\infty)}$ follows immediately from its definition.

Also, since A is unital, we have by definition that $\mathcal{A}_{\mu,D}^{(n)}$, and therefore $\mathcal{A}^{(n)}$ are also unital. Therefore the maps $\mathcal{A}^{(n)} \to \mathcal{A}^{(n-1)}$ will be essential.

To finish the proof we only need to observe that by the construction $[(E_\omega, \mathfrak{b}(D_\omega))] = [(E, F)]$. □

Lemma 2.1.13. *In the conditions of Lemma 2.1.12 one may replace analytic fréchetization with differential one.*

Proof. We construct the system of subalgebras in the same way as in Lemma 2.1.12. We form a system of (semi)norms $\{ {}_1\|\cdot\|, {}_1\|\cdot\|'_1, {}_1\|\cdot\|'_2, \dots \}$, where ${}_m\|\cdot\|'_n$ are as in Lemma 2.1.11. We would like to show that this system is equivalent to a differential seminorm. All the conditions of differential seminorm are easy to check. We shall verify only the third one. Indeed

$$\begin{aligned} {}_1\|ab\|'_n &\leq \sup_{\Omega}\ {}_1\|ab\|_{\mu,n,D_\omega} \\ &\leq \sup_{\Omega} \sum_{k=0}^{n} {}_1\|a\|_{\mu,k,D_\omega}\ {}_1\|b\|_{\mu,n-k,D_\omega} \\ &\leq \sum_{k=0}^{n} (\sup_{\Omega}\ {}_1\|a\|_{\mu,k,D_\omega})(\sup_{\Omega}\ {}_1\|b\|_{\mu,n-k,D_\omega}) \\ &= \sum_{k=0}^{n} {}_1\|a\|'_k\ {}_1\|b\|'_{n-k} \end{aligned}$$

Thus, by Theorem 1.1.23, the algebras $\mathcal{A}_1^{(n)}$ are pre-C^*-albebras, and we only need to state that the algebras $\mathcal{A}^{(n)}$ coincide with $\mathcal{A}_1^{(n)}$ as subalgebras of A. One may also appeal to Remark 1.1.24. □

The Theorem 2.1.7 then becomes an easy corollary of the Lemmas 2.1.12 or 2.1.13.

Corollary 2.1.14 (Proof of Theorem 2.1.7)**.** *Proof.* Indeed, let Λ be the set of (isomorphism classes) of C^*-algebras. For any $B_\lambda \in \Lambda$ choose a set Ω_λ, consisting of the KK-cycles $(E_{\lambda_\omega}, D_{\lambda_\omega})$, such that the map $\Lambda \to \mathrm{KK}(A, B_\lambda)$, given by taking the homotopy class, is surjective. Then $\Omega := \bigcup_{\lambda\in\Lambda} \Omega_\lambda$ is a set. Applying the Lemma 2.1.12 or 2.1.13 we obtain the desired result. □

Remark 2.1.15. Observe that in the Lemmas 2.1.10 and 2.1.11 we haven't imposed the unitality condition on the algebra A. This condition was imposed in the Theorem 2.1.7 for a single purpose: namely, we need to show that the maps $\mathcal{A}^{(n)} \to \mathcal{A}^{(n-1)}$ are essential. It is quite likely that there may be imposed some additional condition on the fréchetization μ which will guarantee the essentiality of the homomorphism $\mathcal{A}^{(n)} \to A$ for nonunital C^*-algebras.

Moreover, we shall need the condition of the essentiality of the map $\mathcal{A}^{(n)} \to A$ only for the construction of inner KK-product.

We should also note that if the algebra A is unital, then we may use the Theorem 1.2.9 instead of Theorem 1.2.8. In this case the algebras $\mathcal{A}_1^{(n)}$ will be operator algebras with involution, and so we may just put $\mathcal{A}^{(n)} := \mathcal{A}_1^{(n)}$.

Remark 2.1.16. As it has been stated in Remark 2.1.5, we may somewhat weaken the definition of smooth systems fréchetizations, replacing operator algebras with completely isometric involutions with operator pseudoalgebras with pseudoinvolutions. The Theorem 2.1.7 will hold also in this framework, if, however, we demand that

- for given μ the algebras $\mathcal{A}_{\mu,D}^{(n)}$ are operator pseudoalgebras with uniformly bounded multiplication map, that is, for all $n \in \mathbb{N}$ there exist such positive numbers $x_{n,\min}$ and $x_{n,\max}$ independent of D, that $x_{n,\min} \leq \|m\|_{cb} \leq x_{n,\max}$ for any cycle (E, D), where by m we mean the multiplication map $m\colon \mathcal{A}_{\mu,D}^{(n)} \tilde{\otimes} \mathcal{A}_{\mu,D}^{(n)} \to \mathcal{A}_{\mu,D}^{(n)}$;
- an analogous property of uniform boundedness holds for pseudo-involution maps $*\colon \mathcal{A}_{\mu,D}^{(n)} \to \mathcal{A}_{\mu,D}^{(n)}$.

These two conditions will ensure that the algebra $\mathcal{A}_1^{(n)}$ which we construct in the Lemma 2.1.11 is indeed an operator pseudoalgebra with pseudo-involution. However, since in case we consider smooth systems consisting of seudoalgebras, we may just put $\mathcal{A}^{(n)} = \mathcal{A}_1^{(n)}$, saving on the complications that we have had in Lemma 2.1.12.

2.1.3 A "Doing It Wrong" Example

We would like to illustrate the Theorem 2.1.7 with some more detailed examples that will show, that it is in principle an "existence" result. The first two are dedicated to showing that the unbounded operators generated by Lemma 2.1.10 may be very different from the standard differentiation operators.

Example 2.1.17. We consider the simplest case, namely, the unit circle. More precisely, we take a Hilbert space $\ell^2(\mathbb{Z})$, where the elements of the basis correspond to the Fourier functions o the circle $\underline{e}_k(x) = \exp(ikx)$ for $x \in [0, 2\pi)$ and $k \in \mathbb{Z}$. Respectively, the Fourier functions $\{e_k(x) := \exp(ikx)\}$ themselves act on this space as "shift by k" operators and the algebra of continuous functions on a circle with supremum norm arises as a norm completion of the algebra generated by these operators with respect to the norm in $\mathbb{B}(\ell^2(\mathbb{Z}))$. Thus we obtain that $\ell^2(\mathbb{Z})$ becomes a $(C(S^1), \mathbb{C})$-bimodule.

We take the Dirac operator $\not{D}$ on S^1, which coincides in our case with the usual differential operator $-i\frac{\partial}{\partial x}$. The Fredholm operator

$$F\colon (\dots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \xi_2, \dots) \mapsto (\dots, -\xi_{-2}, -\xi_{-1}, \underbrace{0}_{0\text{'th place}}, \xi_1, \xi_2, \dots)$$

is then a compact perturbation of $\mathfrak{b}(\not{D})$.

Now, starting with F, we are going to construct an unbounded regular operator on $\ell^2(\mathbb{Z})$, following precisely the recipe of Lemma 2.1.10.

Let $a_1 = 1$, $a_2 = e_1$, $a_3 = e_{-1}$, $a_4 = e_2$ and so on. We want construct an approximate unit $\{u_j\}$ such that $d_j = u_{j+1} - u_j$ will satisfy the properties of the one in theorem, i.e. $\|[F; a_j]d_j\| \leq 2^{-j^2}$ and $[a_j, d_j] \leq 2^{-j^2}$. First observe that since $a_1 = 1$ we automatically have that

$$[F; a_1]d_1 = [F; 1]d_1 = 0$$
$$[d_1, 1] = 0$$

because 1 commute with all the other operators.

Then we calculate

$$[F;\exp(ikx)](\dots,\xi_i,\dots) = (\dots,0,\underbrace{\xi_{-k}}_{0\text{'th place}},2\xi_{-k+1},2\xi_{-k+2},\dots,2\xi_{-1},\xi_0,0,\dots)$$

and analogously

$$[F;\exp(-ikx)](\dots,\xi_i,\dots) = (\dots,0,\xi_0,2\xi_1,2\xi_2,\dots,2\xi_{k-1},\underbrace{\xi_k}_{0\text{'th place}},0,\dots)$$

for $k \geq 1$. In particular, it follows that

$$[F;a_2]\xi = (\dots,0,\underbrace{\xi_{-1}}_{0\text{'th place}},\xi_0,0\dots)$$

We may now assume $u_1 = 0$, and in order to fulfill all the conditions for $j = 2$ we put:

$$u_2 := \operatorname{diag}(\dots,0,1\cdot 2^{-4},2\cdot 2^{-4},\dots,15\cdot 2^{-1},1\underbrace{1}_{0\text{'th place}},1,15\cdot 2^{-4},\dots,2^{-4},0,\dots)$$

and

$$\begin{aligned} u_3 = \operatorname{diag}(&\dots,0,2^{-9},2\cdot 2^{-9},\dots,511\cdot 2^{-9},511\cdot 2^{-9},\\ &\underbrace{1,\dots,1}_{35\text{ times}},511\cdot 2^{-9},\dots,2^{-9},0,\dots)\end{aligned}$$

where u_3 is symmetric from 0'th position as well as u_2. It could be checked directly that $[F;a_2]d_2 \equiv 0$ and $\|[a_2;d_2]\| = \frac{1}{16}$ (because the vector $[a_2;d_2]\xi$ has either 0 or $2^{-9}\xi_i$ or $2^{-4}\xi_i$ on $i+1$'th place).

The choice of u_4 is somewhat more complicated, because we need it for the construction of both $d_3 = u_4 - u_3$ and $d_4 = u_5 - u_4$. Recall that by our definition $a_4 = \exp(2\pi i \cdot 2x)$, so that a_4 acts as "shift by 2" operator. To fulfill the conditions, we put:

$$\begin{aligned} u_4 := \operatorname{diag}(&\dots,0,2^{-25},2^{-25},2\cdot 2^{-25},2\cdot 2^{-25},\dots,33554431\cdot 2^{-25},33554431\cdot 2^{-25},\\ &\underbrace{1,\dots,1}_{1059\text{ times}},33554431\cdot 2^{-25},33554431\cdot 2^{-25},\dots,2^{-25},2^{-25},0,\dots)\end{aligned}$$

and one may again check that $u_3u_4 = u_3$, $[F;a_3]d_3 = 0$ and $\|[a_3,d_3]\| = 2^{-9}$.

Taking into consideration the same observations, for a general index j we write

$$\begin{aligned} u_j = \operatorname{diag}(&\dots,0,\underbrace{2^{-(j+1)^2},\dots,2^{-(j+1)^2}}_{t_j\text{ times}},\underbrace{2\cdot 2^{-(j+1)^2},\dots,2\cdot 2^{-(j+1)^2}}_{t_j\text{ times}},\dots\\ &\dots,\underbrace{(2^{(j+1)^2}-1)\cdot 2^{-(j+1)^2},\dots,(2^{(j+1)^2}-1)\cdot 2^{-(j+1)^2}}_{t_j\text{ times}},\underbrace{1,1,\dots,1,1,}_{1\text{ on all nonzero places of }u_{j-1}}\\ &\underbrace{(2^{(j+1)^2}-1)\cdot 2^{-(j+1)^2},\dots,(2^{(j+1)^2}-1)\cdot 2^{-(j+1)^2}}_{t_j\text{ times}},\dots,\underbrace{2^{-(j+1)^2},\dots,2^{-(j+1)^2}}_{t_j\text{ times}},0,\dots)\end{aligned}$$

where t_j equals to $\frac{j}{2}(\frac{j}{2}-1)$ when j is even and $\frac{j-1}{2}$ when it is odd.

We may finally notice, that u_j constructed this way is an approximate unit, commuting with F and quasicentral for $C(S^1)$.

As in the Theorem 2.1.10, we construct an unbounded function

$$r = \sum_{j=1}^{\infty} 2^j d_j$$

and put $D = Fr$. By the construction, $\mathfrak{b}(D)$ and $\mathfrak{b}(\not{D})$ are both compact perturbations of F. To feel the difference, we look at the spectrum of these operators.

Recall [34] that the *classical dimension* of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is defined as follows. For an operator D we take its resolvent $\mathfrak{r}(D) = (1 + D^2)^{-\frac{1}{2}}$ which is a compact operator on $\mathcal{H}$ and consider the sequence $\{\mu_n\}$ of the eigenvalues of $\mathfrak{r}(D)$, such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq$. The classical dimension of the spectral triple is then the minimal positive number α for which $\mu_n \sim O(n^{-\frac{1}{\alpha}})$. Thus, for a circle with a standard differential operator the classical dimension equals to 1.

Now consider the resolvent of the operator D defined above. Since it has a diagonal form in the basis given by the Fourier functions, one can easily pick up the eigenvalues of the resolvent of D. Namely, $\mu_0 = 1/\sqrt{2}$ and

$$\mu_{2k} = \mu_{2k+1} = (1 + \lambda_k^2)^{-\frac{1}{2}}$$

where λ_k is the value of the entry standing k positions away from the 0'th position. Here we used the fact that the unbounded operator r has entries that are symmetric from 0'th position, and so for $D = Fr$ one has $\lambda_k^2 = \lambda_{-k}^2$.

We are going to estimate the values of λ_k. Observe first that $d_{j+2}\underline{e}_k = 0$ for all such k that $d_j\underline{e}_k \neq 0$, and this holds for all $j \in \mathbb{N}$. Thus, for every λ_k we have that

$$|\lambda_k| \leq 2^j + 2^{j+1} = 3 \cdot 2^j$$

where j is such a number that $d_j\underline{e}_k \neq 0$ and $d_{j+1}\underline{e}_k \neq 0$. Hence, by a very rough estimation where we, for instance, ignore that the numbers t_j are in general sufficiently greater then 1, we obtain that for $k \leq 2^{j^2}$ one has $|\lambda_k| \leq 3 \cdot 2^j$. Thus we have that

$$\mu_k \geq (1 + 6 \cdot 2^{2\sqrt{\log_2(k/2)}})^{-\frac{1}{2}}$$

and hence may finally conclude that

$$2^{(\log_2 k)^{-\frac{1}{2}}} = O(\mu_k)$$

But there is no such $\alpha \geq 0$, for which $2^{(\log_2 k)^{-\frac{1}{2}}}$ could have been an $O(n^{-\frac{1}{\alpha}})$. Hence, informally speaking, the spectral triple on a circle with D as a differential operator would have the "dimension" infinity.

It is important to point out, that the class of the cycle $(\ell^2(\mathbb{Z}), D)$ with D constructed above is a generator of $K_1(S^1)$, the odd K-homology group of the circle. The generator of $K_0(S^1)$ may be obtained by taking the cycle $(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}), \left(\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}\right))$. Denote these two cycles by x and y, and let $-x = (\ell^2(\mathbb{Z}), -D)$ and $-y = (-(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})), -\left(\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}\right))$, where by $-(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}))$ we mean the space $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ with grading given by the unitary $-\gamma$, where $\gamma := \mathrm{diag}(\mathrm{Id}_{\ell^2(\mathbb{Z})}, -\mathrm{Id}_{\ell^2(\mathbb{Z})})$ was the grading on $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$. Then the cycles of the form $k_1 x \oplus k_2 y$ where $k_1, k_2 \in \mathbb{Z}$ generate the whole group $K_0(S_1) \oplus K_1(S^1)$. Thus, for a commutator bounded analytic fréchetization μ the smooth system $\mathscr{A}_{\mu,D}$ may be taken as the one constructed in Lemma 2.1.11. Hence, the system $\mathscr{A}_{\mu,D}$ supports all the K-homology of the circle, and yet the functions in this system differ drastically from the C^∞.

To make the difference between the smooth system generated by D and by $-i\frac{\partial}{\partial x}$ even more apparent, we include the following example:

Example 2.1.18. Let μ be a commutator bounded fréchetization. Consider the system of functions $\{l_k(x)\}$ on $C(S^1)$, $k \in \mathbb{Z}$, where

$$l_0(x) \equiv 1$$

and, subsequently,

$$l_k\left(\frac{m}{4k}\right) = \exp\left(\frac{\pi i m}{2}\right)$$

for $m = 0, 1, \ldots, 4k$ and are linear between $\frac{m}{4k}$ and $\frac{m+1}{4k}$ for $m = 0, 1, \ldots, 4k-1$. Informally, the functions $l_k(x)$ could be regarded as a "linearization" of the Fourier functions. The system $\{l_k(x)\}$ is total for $C(S^1)$ (one may check directly, that for every $\varepsilon > 0$ there exist $N = N(\varepsilon) \in \mathbb{N}$, and the sequence $\{c_k\}$, $c_k \in \mathbb{C}$, such that $\|\exp(2\pi i x) - \sum_{k=1}^{N} c_k l_k(x)\| < \varepsilon$).

If we consider the operator F form the previous example and take $\{l_k\}$ as a total system of $C(S^1)$, then, taking a subsequence of the approximate unit u_k as we have constructed it above, by Theorem 2.1.7 we may construct an operator D, such that $\mathfrak{b}(D) = F$ and $l_k \in \mathcal{A}^{(j)}_{\mu,D}$ for all $j \in \mathbb{N}$. Now, for an arbitrary $n \in \mathbb{N}$ put

$$s_m = \|l_{2^m} + l_{-2^m}\|_{\mu,n,D}$$

Then the series

$$w_M(x) = \sum_{m=1}^{M} \frac{l_{2^m}(x) + l_{-2^m}(x)}{2^m s_m}$$

converges with respect to the norm $\|\cdot\|_{\mu,n,D}$ to a function $w(x) \in \mathcal{A}^{(n)}_{\mu,D}$, so that by our definition the function w is to be considered as n-smooth. However, $w(x)$ is nothing else then a slightly modified Weierstrass Sew, a function that is not differentiable at any point (of the unit interval). Thus the algebra of functions, smooth with respect to a chosen differential operator may be much wider then the classical one.

Remark 2.1.19. The Examples 2.1.17 and 2.1.18 may be generalized to the case of multi-dimensional tori (including noncommutative ones). However, we have chosen the case of circle because it is the simplest, and, therefore, is from one hand easy to grasp and from the other hand allows us to make the calculations not to be ridiculously complicated.

The consequences of these examples are twofold. From one hand, we see that the procedure in Lemma 2.1.10 will not give us habitual operators even in the most simple cases. From the other hand, we see that $\mathcal{A}^{(n)}_{\mu,D}$ appears to be sufficiently larger than $\mathcal{A}^{(n)}_{\mu,\not{D}}$, so that we may expect that the familiar Dirac-type operators yield smooth systems having interesting KK theory.

2.1.4 Standard Fréchetizations

In this section we are going to describe two fréchetizations that may be regarded as standard ones. They were chosen this way because they are the most informative at the current moment. The first one, the *commutator fréchetization* is more easily defined and so may be used for pure theoretical means. The second one, which we call the *Mesland fréchetization*, is more elaborated and is suited to deal with the tasks coming out from differential geometry ant theory of spectral triples.

Commuator Fréchetization

We start with the most basic fréchetization. This one will be called *commutator* or *ad-fréchetization* and the algebras generated by it will be denoted by $\mathcal{A}^{(n)}_{\mathrm{ad},D}$.

Let

$$\Theta^1_D(a) := \begin{pmatrix} a & 0 \\ [D;a] & \gamma a \gamma \end{pmatrix}$$

as an operator on $E^{\oplus 2}$. Here γ is a grading operator on E. Analogously we set

$$\Theta^{n+1}_D(a) := \begin{pmatrix} \Theta^n_D(a) & 0 \\ [D;\Theta^n_D(a)] & \gamma_n \Theta^{n-1}_D(a) \gamma_n \end{pmatrix}$$

on $E^{\oplus 2^{n+1}}$, where we abusively denote by D the operator $\mathrm{diag}(\underbrace{D,\dots,D}_{2^{n-1}\text{ times}})$ on $E^{\oplus 2^{n-1}}$, and γ_n is the natural grading on $E^{\oplus 2^{n+1}}$, which is inductively defined as

$$\gamma_m = \begin{pmatrix} \gamma_{n-1} & 0 \\ 0 & -\gamma_{n-1} \end{pmatrix}$$

We also denote by Θ^0_D the map $a \mapsto a$.

It may be checked directly that

$$\Theta^n_D(ab) = \Theta^n_D(a)\Theta^n_D(b)$$

and that there exists a unitary operator $u_n \in \mathrm{End}^*_B(E^{\oplus 2^n})$ such that

$$u_n(\Theta^n_D(a))^* u_n^{-1} = \Theta^n_D(a^*) \tag{2.3}$$

Now, define an algebra

$$\mathcal{A}^{(n)}_{\mathrm{ad},D} := \{a \in A \mid \mathrm{ad}^k_D(a) \text{ extends to bounded on } E \text{ for all } k = 1, \dots, n\}$$

This is an algebra, which is complete with respect to the norm $\|\cdot\|_{\mathrm{ad},n,D} := \|\Theta^n_D(a)\|$. By the construction we have an estimate

$$\|a\|_{\mathrm{ad},n,D} \leq 2^n \max\{\|a\|, \|\mathrm{ad}_D(a)\|, \dots, \|\mathrm{ad}^n_D(a)\|\} \tag{2.4}$$

We endow $\mathcal{A}^{(n)}_{\mathrm{ad},D}$ with the operator algebra structure given by the representation

$$\Theta^n_D(a) \colon a \mapsto \Theta^n_D(a)$$

The invoulution on A induces an operator algebra involution on $\mathcal{A}^{(n)}_{\mathrm{ad},D}$. Indeed, according to the equation 2.3, the involution on $\mathcal{A}^{(n)}_{\mathrm{ad},D}$ is isometric. To show that it is completely isometric, we use the following observation, that will be referred to as standard throughout the section. We point to the fact that for any $n \in \mathbb{N}$ by the construction of $\Theta^n_D(\cdot)$ for all $m \in \mathbb{N}$ there exists a unitary operator $v_{m,n} \in M_{m\cdot 2^{n+1}}(\mathcal{M}(A))$ with 0 or Id_E such that we have

$$(\Theta^n_D(a_{jk}))_{jk} = v_{m,n}\Theta^n_{\mathrm{diag}_m(D)}((a_{jk})_{jk})v^{-1}_{m,n}$$

Therefore

$${}_m\|(a_{jk})_{jk}\|'_{\mathrm{ad},n,D} = {}_1\|(a_{jk})_{jk}\|_{\mathrm{ad},n,\mathrm{diag}_m(D)}$$

and we have established a completely isometric isomorphism

$$M_m(\mathcal{A}^{(n)}_{\mathrm{ad},D}) \approxeq (M_m(A))_{\mathrm{ad},\mathrm{diag}_m(D)} \tag{2.5}$$

Now, we see that

$$\|(\Theta^n_D(a^*_{kj}))_{jk}\| = \|\Theta^n_{\mathrm{diag}_m D}((a^*_{kj})_{jk})\| = \|\Theta^n_{\mathrm{diag}_m D}((a_{jk})_{jk})\| = \|(\Theta^n_D(a_{jk}))_{jk}\|$$

which means that

$${}_m\|(a_{jk})^*_{jk}\|_{\mathrm{ad},n,D} = {}_m\|(a_{jk})_{jk}\|_{\mathrm{ad},n,D}$$

and so the involution is indeed completely isometric.

The operator algebras $\mathcal{A}^{(n)}_{\mathrm{ad},D}$ are stable under the holomorphic functional calculus on A. Indeed, in case when $\mathcal{A}^{(n)}_{\mathrm{ad},D}$ is dense in A, for all $a \in \mathcal{A}^{(n-1)}_{\mathrm{ad},D}$ such that $\|a\|_{\mathrm{ad},n-1,D} < 1$ we have that

$$\begin{aligned}
\limsup_{m\to\infty} \frac{\ln\|a^m\|_{\mathrm{ad},n,D}}{m} &= \limsup_{m\to\infty} \frac{\ln\|\Theta^n_D(a^m)\|}{m} \\
&\leq \limsup_{m\to\infty} \frac{\ln(\|\Theta^{n-1}_D(a^m)\| + \|[D, \Theta^{n-1}_D(a^m)]\|)}{m} \\
&\leq \limsup_{m\to\infty} \frac{\ln(1 + m\|\Theta^{n-1}_D(a)\|)}{m} \\
&\leq \limsup_{m\to\infty} \left(\frac{\ln m}{m} + \frac{\ln(1 + \|\Theta^{n-1}_D(a))\|}{m}\right) \\
&= 0
\end{aligned} \tag{2.6}$$

Thus the norm $\|a\|_{\text{ad},n,D}$ is analytic with respect to $\|a\|_{\text{ad},n-1,D}$, and so $\mathcal{A}^{(n)}_{\text{ad},D}$ are stable under the holomorpic functional calculus on A.

By the construction we have that there is an inclusion $\mathcal{A}^{(n+1)}_{\text{ad},D} \hookrightarrow \mathcal{A}^{(n)}_{\text{ad},D}$, which is completely contractive. Indeed, we have that

$$\begin{aligned}
\|a\|_{\text{ad},n+1,D} &= \|\Theta^{n+1}(a)\| \\
&= \left\| \begin{pmatrix} \Theta^n_D(a) & 0 \\ [D;\Theta^n_D(a)] & \gamma_n \Theta^{n-1}_D(a)\gamma_n \end{pmatrix} \right\| \\
&\geq \|\Theta^n_D(a)\| \\
&= \|a\|_{\text{ad},n,D}
\end{aligned} \tag{2.7}$$

so that the inclusion is contractive. To see that it is completely contractive, we use the standard agruement about the unitary $v_{m,n}$. Thus, according to the estimate 2.7 we have that

$$\|\Theta^{n+1}_{\text{diag}_m(D)}((a_{jk})_{jk})\| \geq \|\Theta^n_{\text{diag}_m(D)}((a_{jk})_{jk})\|$$

and therefore

$${}_m\|(a_{jk})_{jk}\|'_{\text{ad},n+1,D} \geq {}_m\|(a_{jk})_{jk}\|_{\text{ad},n,D}$$

Thus we have shown that the inclusion $\mathcal{A}^{(n+1)}_{\text{ad},D} \to \mathcal{A}^{(n)}_{\text{ad},D}$ is completely contractive with respect to the norms ${}_m\|\cdot\|_{\text{ad},n,D}$.

Putting everything together, we see that

- Since $\mathcal{A}^{(n)}_{\text{ad},D}$ are operator algebras with compleltey isometric involution induced by the involution on $\mathcal{A}$ the commutator fréchetization is indeed a fréchetization.
- According to the inequality 2.4 it is commutator bouded.
- By the equation 2.6 it is analytic.
- It is also obvious that if A is unital, then $\Theta^n_D(1) = \text{diag}_{2^n} 1$, so that the fréchetization preserves constants and $\|1\|_{\text{ad},n,D} = 1$.

Therefore the commutator fréchetization satisfies the conditions of the Theorem 2.1.7. Therefore the apllication of this fréchetiztion with unital separable C^*-algebras will not lead to loss of information of their KK-theory.

We also see that if $a \in \mathcal{A}^{(n)}_{\text{ad},D}$, then $a\colon \text{Dom} D_n \to \text{Dom} D_n$. For that it is enough to show that $D^n(ah)$ for all $h \in \text{Dom} D^n$. But we have that

$$D^n ah = \sum_{k=0}^{n} \pm \binom{n}{k} \text{ad}^k_D(a) D^{n-k} h$$

where $\binom{n}{k}$ are binomial coefficients. Since $\text{ad}^k_D(a) \in \text{End}^*_A(E)$ and $h \in \text{Dom}(D^n)$ all the summands

$$\pm \binom{n}{k} \text{ad}^k_D(a) D^{n-k} h$$

above are defined. Hence $D^n ah$ may be represented as a finite sum of vectors in E, and so $D^n ah \in E$.

The commutator fréchrtization is quite easy to grasp, but the number of cases it may be applied to is not satisfactory. For instance, if we take the Dirac operator $\not{D}$ on a torus of dimension $m \geq 2$, then for all nonconstant functions a in the algebra $C^2(\mathbb{T}^m)$ we may see that the already the operators $\mathrm{ad}^2_{\not{D}}(a)$ are not bounded. The same observation holds for noncommutative tori with the Dirac operator introduced as in [23, Sect. 12.3],[34]. Therefore the commutator fréchetization may only be used for theoretical means and as internals steps in calculations (like the ones we have in the next subsection), but not for a work with geometrical objects.

Mesland Fréchetization

The process that allows us to include the standard Dirac-type operators on (noncommutative) manifolds was developed by Mesland in [28]. We shall outline the process of construction and the main properties of this fréchetization here.

For an unbounded even (A, B)-KK-bimodule (E, D) we let

$$\pi^1_D := \begin{pmatrix} a & 0 \\ [D; a] & \gamma a \gamma \end{pmatrix}$$

where γ is the grading on E.

There is a representation

$$\begin{array}{rcl} \mathcal{A}^{(1)}_D & \to & \mathrm{End}^*_B(\mathfrak{G}(D)) \\ a & \mapsto & p_D \pi^1_D(a) p_D \end{array}$$

This is an algebra homomorphism due to the identity $p_D \pi^1_D(a) p_D = \pi^D_1 p_D$. Denote $p^\perp_D = 1 - p_D$. We obtain that

$$\begin{array}{rcl} \mathcal{A}^{(1)}_D & \to & \mathrm{End}^*_B(v\mathfrak{G}(D)) \\ a & \mapsto & (1-p_D)\pi^1_D(a)(1-p_D) \end{array}$$

is also a homomorphism. We may now define the map

$$\begin{array}{rll} \theta^1_D\colon & \mathcal{A}^{(1)}_D & \to M_2(\mathrm{End}^*_B(E)) \\ & a & \mapsto p_D \pi^1_D(a) p_D + (1-p_D)\pi^1_D(a)(1-p_D) \end{array}$$

Let $\mathcal{A}^{(1)}_D$, π^1_D and θ^1_D be as above. For $n \geq 1$ we abusively denote by D the operator $\mathrm{diag}(D, \ldots, D)$ on $\bigoplus_{j=1}^n E$ and by p_D its Woronowicz projection. There is a natural grading on $\bigoplus_{j=1}^{2^{n+1}} E$, which is defined inductively by

$$\gamma_{i+1} := \begin{pmatrix} \gamma_i & 0 \\ 0 & -\gamma_i \end{pmatrix}$$

Now, we inductively define the maps

$$\begin{aligned}
\pi_D^{n+1}\colon A \supseteq \mathrm{Dom}\pi_D^{n+1} &\to M_{2^{n+1}} \in \mathrm{End}_B^*(E)\\
a &\mapsto \begin{pmatrix} \theta_D^n(a) & 0 \\ [D;\theta_D^n(a)] & \gamma_n\theta_D^n(a)\gamma_n \end{pmatrix}\\
\theta_D^{n+1}\colon A \supseteq \mathrm{Dom}\theta_D^{n+1} &\to M_{2^{n+1}}(\mathrm{End}_B^*(E))\\
a &\mapsto p_{D,n+1}p_{D,n}\pi_{n+1}^D(a)p_{D,n}p_{D,n+1}\\
&\quad + p_{D,n+1}^\perp p_{D,n}^\perp \pi_{n+1}^D(a)p_{D,n}^\perp p_{D,n+1}^\perp
\end{aligned}$$

where we abusively write $p_{D,n}$ and $p_{D,n}^\perp$ for $\mathrm{diag}(p_{D,n}, p_{D,n})$ and $\mathrm{diag}(p_{D,n}^\perp, p_{D,n}^\perp)$ respectively.

The smooth system is then defined as

$$\mathcal{A}_{mes,D}^{(n+1)} := \{a \in \mathcal{A}_{mes,D}^{(n)} \mid [D, \theta_D^n(a)] \text{ extends to a element of } \mathrm{End}_B^* E^{\oplus 2^n}\}$$

This algebra is represented as an operator algebra on $\bigoplus_{k=0}^n \bigoplus_{j=1}^{2^k} E$ by the map $a \mapsto \bigoplus_{k=0}^n \pi_D^n(a)$. Clearly, the inherited operator norm on $\mathcal{A}_{mes,D}^{(n)}$ then equals to

$$\|\cdot\|_{mes,n,D} = \max_{k=0}^{n} \|\pi_D^n(\cdot)\|$$

It has been proved in [28], that in the case when D is selfadjoint $\mathcal{A}_{mes,D}^{(n)}$ are operator algebras with a completely isometric involution induced by involution on A. It is also shown that the norm $\|\cdot\|_{mes,n+1,D}$ is analytic with respect to $\|\cdot\|_{mes,n,D}$. Therefore, the system $\mathscr{A}_{mes,D}$ is indeed a smooth system. By the construction, the order of the system is at least 1, and the existence of the smooth subalgebras of greater order should be checked individually.

For each D there is a completely bounded inclusion $\mathcal{A}_{\mathrm{ad},D}^{(n)} \hookrightarrow \mathcal{A}_{mes,D}^{(n)}$ for all n. Indeed, suppose that the algebra $\mathcal{A}_{ad,D}^{(n)}$ is dense in A, so that the operators $\mathrm{ad}_D^n(a)$ are defined for a dense subset of A. We observe that by definition $\pi_D^1(a) = \Theta_D^1(a)$. Next

$$\begin{aligned}
\pi_D^2(a) &= \begin{pmatrix} \theta_D^1(a) & 0 \\ [D;\theta_D^1(a)] & \gamma_2\theta_D^1(a)\gamma_2 \end{pmatrix}\\
&= \begin{pmatrix} p_{1,D}\Theta_D^1(a)p_{1,D} + p_{1,D}^\perp\Theta_D^1(a)p_{1,D}^\perp & 0 \\ [D; p_{1,D}\Theta_D^1(a)p_{1,D} + p_{1,D}^\perp\Theta_D^1(a)p_{1,D}^\perp] & \gamma_1(p_{1,D}\Theta_D^1(a)p_{1,D} + p_{1,D}^\perp\Theta_D^1(a)p_{1,D}^\perp)\gamma_1 \end{pmatrix}\\
&= p_{1,D}\begin{pmatrix} \Theta_D^1(a) & 0 \\ [D,\Theta_D^1(a)] & \gamma_1\Theta_D^1(a)\gamma_1 \end{pmatrix} p_{D,1} + p_{D,1}^\perp \begin{pmatrix} \Theta_D^1(a) & 0 \\ [D,\Theta_D^1(a)] & \gamma_1\Theta_D^1(a)\gamma_1 \end{pmatrix} p_{D,1}^\perp\\
&= p_{D,1}\Theta_D^2(a)p_{D,1} + p_{D,1}^\perp\Theta_D^2(a)p_{D,1}^\perp
\end{aligned}$$

since, by definition $p_{D,1}$, γ_1 and D commute. Therefore

$$\begin{aligned}
\theta_D^2(a) &= p_{D,2}p_{D,1}\pi_2^D(a)p_{D,1}p_{D,2} + p_{D,2}^\perp p_{D,1}^\perp\pi_2^D(a)p_{D,1}^\perp p_{D,2}^\perp\\
&= p_{D,2}p_{D,1}\Theta_2^D(a)p_{D,1}p_{D,2} + p_{D,2}^\perp p_{D,1}^\perp\Theta_2^D(a)p_{D,1}^\perp p_{D,2}^\perp
\end{aligned}$$

Continuing this process inductively, we obtain that

$$\pi_D^n(a) = \prod_{k=1}^n p_{k,D}\Theta_D^n(a)\prod_{k=1}^n p_{n-k+1,D} + \prod_{k=1}^n p_{k,D}^{\perp}\Theta_D^n(a)\prod_{k=1}^n p_{n-k+1,D}^{\perp}$$

Thus, we have that

$$\|\pi_D^n(a)\| \leq \|\Theta_D^n(a)\|$$

and so

$$\|a\|_{mes,n,D} = \max_{k=0}^n\{\|\pi_D^n(a)\|\} \leq \max_{k=0}^n\{\|\Theta_D^n(a)\|\} = \|\Theta_D^n(a)\| = \|a\|_{\mathrm{ad},n,D}$$

Thus there is a contractive inclusion $\mathcal{A}_{mes,D}^{(n)} \to \mathcal{A}_{\mathrm{ad},D}^{(n)}$.

As for the complete contractiveness, we may again use the standard arguement. Observe that for each $m \in \mathbb{N}$ there is a unitary operator $v_{m,n} \in M_{m\cdot 2^{n+1}}(\mathcal{M}(A))$, acting as a permutation operation (so it actually has only 0 and 1 entries), such that

$$v_{m,n}(\Theta_D^n(a_{kl}))_{kl}v_{m,n}^{-1} = \Theta_{\mathrm{diag}_m D}^n((a_{kl}))$$

Moreover, we have that $v_{m,n}\mathrm{diag}_m p_D^n v_{m,n}^{-1} = p_{\mathrm{diag}_m(D)}$, and so, applying the same operator $v_{m,n}$ we obtain that

$$v_{m,n}(\pi_D^n(a_{kl}))_{kl}v_{m,n}^{-1} = \pi_{\mathrm{diag}_m D}^n((a_{kl}))$$

Hence

$$\begin{aligned}
{}_m\|(a_{kl})\|_{mes,n,D} &= \max_{j=0,\dots,n}\|(\pi_D^j(a_{kl}))_{kl}\| \\
&= \max_{j=0,\dots,n}\|v_{m,n}(\pi_D^j(a_{kl}))_{kl}(\pi_D^j(a_{kl}))_{kl}v_{m,n}^{-1}\| \\
&= \max_{j=0,\dots,n}\|\pi_{\mathrm{diag}_m D}^j((a_{kl}))\| \\
&\leq \max_{j=0,\dots,n}\|\Theta_{\mathrm{diag}_m D}^j((a_{kl}))\| \\
&= \max_{j=0,\dots,n}\|v_{m,n}^{-1}\Theta_{\mathrm{diag}_m D}^j((a_{kl}))v_{m,n}\| \\
&= \max_{j=0,\dots,n}\|(\Theta_D^j(a_{kl}))_{kl}\| \\
&= {}_m\|(a_{kl})\|_{\mathrm{ad},n,D}
\end{aligned}$$

and we may use the previous observation.

These estimates show that *mes* fréchetization is also commutator bounded, and so we may also apply Theorem 2.1.7 to it.

The Mesland fréchetization may have useful applications in noncommutative geometry. It has been shown in [28] that the algebras $\mathcal{A}_{mes,D}^{(\infty)}$ satisfy the notion of smoothness, fomulted via the Regularity condition introduced by Alain Connes. We recall that theregularity means that for all $a \in \mathcal{A}_{mes,D}^{(\infty)}$ both a and $[D;a]$ lay in $\cap_{n=1}^{\infty}\delta^n$, where δ is a derivation

$\delta\colon T \mapsto [|D|, T]$. It may also be shown that if $(\mathcal{H}, D)$ are actually a spinor bundle and a Dirac operator on a spin-manifold M, then the order of smoothness of the system $\mathscr{A}_{mes,D}$, induced by operator D on the C^*-algebra $A := C(M)$, is infinite. Therefore Mesland fréchetization seems to be the best candidate to work with KK-theoretical tasks arising in differential geometry.

It has also been shown in[28] that for a selfadjoint regular D the following operator norms are equivalent: ${}_m\|\cdot\|_{mes,n,D}$, ${}_m\|\cdot\|_{mes,n,|D|}$, ${}_m\|\cdot\|_{mes,n,cD}$ and ${}_m\|\cdot\|_{mes,n,D+b}$ for $c \in \mathbb{R}$ and $b \in \mathrm{Sob}^{(n)}_{mes,D}$, $b = b^*$. Therefore the smooth systems $\mathscr{A}_{mes,D}$ and $\mathscr{A}_{mes,cD+b}$ are equivalent.

We recall that in noncommutative geometry the Dirac-type operator is used to define an analogue of noncommutative metric. Following [16], if M is a spin-manifold, $\mathcal{A} := C^\infty(M)$ and $\not{D}$ is a Dirac-type operator on M, then, given two points $p, q \in M$ one may define a distance between p and q by setting.

$$\mathrm{dist}(p,q) := \sup\{|\hat{p}(a) - \hat{q}(a)| \mid a \in \mathcal{A},\ \|[\not{D}, a]\| \le 1\}$$

where $\hat{p}$ and $\hat{q}$ are the characters on the algebra $\mathcal{A}$, corresponding to the points p and q. It has been proved by Alain Connes in [16] that the distance so defined coincides with the distance defined by Riemann metric on the manifold M. We should also note that if we replace $\mathcal{D}$ with cD, then the distance between p and q will extend by factor c^{-1}, and if b is an (odd) operator on a spinor bundle, commuting with the action of a, then $\mathcal{D}$ and $\mathcal{D} + b$ define the same distance.

By these consideration, having an unbounded KK-cycle (E, D), we may consider an unbounded KK-cycle $(E, cD + b)$ as its "linear rescaling". The properties of Mesland fréchetization then tell us, that if $(E, D) \in \Psi^{(n)}_{mes}(\mathscr{A}, B)$ then the same holds for all its "linear rescaling" KK-cycles, i.e. $(E, cD + b) \in \Psi^{(n)}_{mes}(\mathscr{A}, B)$.

It would be interesting to see, what happens when we encounter more complex changes of operators. For instance, if we have two diffeomorphic structures of Riemann manifold on the same smooth topological manifold M, which is spin, then these two structures define two Dirac operators $\not{D}_1$ and $\not{D}_2$. These two Dirac operators may correspond to the metric of a flat torus and the metric induced by embedding the surface of a mug into a 3-dimensional Euclidean space. It seems likely that if E then is a spinor bundle and $\mathscr{A}$ is some smooth system on the algebra $C(M)$, then if $(E, \not{D}_1) \in \Psi^{(n)}_{mes}(\mathscr{A}, \mathbb{C})$ then $(E, \not{D}_2) \in \Psi^{(n)}_{mes}(\mathscr{A}, \mathbb{C})$. If this holds, then we shall be able to consider the system $\mathscr{A}$ as defining a kind of smooth noncommutative topology in a sense that the structure of smooth manifold is identified with all possible smooth Riemann metrics on this manifold. However, this guess needs additional development, so we would not speculate in this direction any further.

2.1.5 Example: Smooth Systems on Noncommutative Tori

The noncommutative tori are one of the simplest object in the C^*-algebra theory. In this subsection we are going to construct a smooth system on noncommutative tori and then compare it to the notion of smooth functions defined for spectral triples ([14], [23]) and the one generated by the construction of Mesland.

By definition [32], the C^*-algebra of an m-dimensional noncommutative torus is a C^*-completion of the involutive associative algebra, generated by unitaries $u_1, u_2, \ldots, u_m$ subject to the relations

$$u_l u_k = e(2\pi i \sigma_{lk}) u_k u_l$$

where $e(t) = e^{2\pi i t}$ and $\sigma := (\sigma_{kl}) \in M_m(\mathbb{R})$ is an antisymmetric matrix. This algebra will be denoted by A_σ.

We employ the notation $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \ldots u_m^{\alpha_m}$ for $\alpha \in \mathbb{Z}^m$. We shall also denote $|\alpha| := \sum_{k=1}^m |\alpha_k|$ and $\|\alpha\| := (\sum_{k=1}^m \alpha_k^2)^{-\frac{1}{2}}$.

Let now B be a C^*-algebra and let (E, D) be an unbounded (A, B)-KK-cycle, such that $\mathrm{ad}_D^n(u_k) < \infty$ for all $k = 1, \ldots, m$. Then, applying the Leibniz rule we obtain that

$$\mathrm{ad}_D(u^\alpha) = \sum_{(l_1, l_2, \ldots l_{|\alpha|})} \mathrm{ad}_D^{l_1}(u_{k(l_1)}^{\pm 1}) \mathrm{ad}_D^{l_2}(u_{k(l_2)}^{\pm 1}) \ldots \mathrm{ad}_D^{l_{|\alpha|}}(u_{k(l_{|\alpha|})}^{\pm 1})$$

Here $l_i \in \mathbb{N} \cup \{0\}$, $\sum_{i=1}^{|\alpha|} l_i = |\alpha|$, and $k(l_i) = j$, $j \in \{1, \ldots m\}$ whenever $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_{j-1}| < i \leq |\alpha_1| + |\alpha_2| + \cdots + |\alpha_j|$ and the sign over u_k is $+$ if $\alpha_k \geq 0$ and $-$ otherwise.

We haven't made any further assumptions on D, and so for the author it is for now possible only to apply the roughest estimate of the norm. Namely, since $\|u^\alpha\|$ are unitaries, we may calculate that

$$\|\mathrm{ad}_D^n(u^\alpha)\| \leq |\alpha|^n \max_{\beta \in \mathbb{Z}^m, |\beta| = n} \prod_{k=1}^m \|\mathrm{ad}_D^{\beta_k}(u_k)\| =: |\alpha|^n K_{\alpha, n, D}$$

This estimate gives us, in particular, that D generates an ∞-smooth system for the commutator fréchetization (and hence also for Mesland fréchetization).

By Lemma 2.1.12 for any set of isomorphism classes of C^*-algebras $\Lambda = \{B_\lambda\}$ we can find a set Ω of unbounded KK-cycles $(E_{\lambda,\omega}, D_{\lambda,\omega})$ that, from one side, the map $\{(E, D)\} \to \mathrm{KK}(A_\sigma, B_\lambda)$ is surjective, and from the other side

$$\sup_{D \in \Omega} K_{\alpha, n, D_{\alpha,\omega}} = K_{\alpha, n} < \infty$$

Thus we are in the conditions of Lemma 2.1.11, and so for any commutator bounded analytic fréchetization μ we may construct algebras $\mathcal{A}_{\sigma,\mu}^{(n)}$, forming an ∞-smooth system $\mathscr{A}_{\sigma,\mu} = \{\mathcal{A}_{\sigma,\mu}^{(n)}\}$.

Now for general $a \in A_\sigma$ we have that

$$\|\mathrm{ad}_D^n(a)\| \leq K_{\alpha, n, D} \sum_{\alpha \in \mathbb{Z}} |c_\alpha| |\alpha^n| \leq K_{\alpha, n} \sum_{\alpha \in \mathbb{Z}} |c_\alpha| |\alpha^n|$$

where we write $a = \sum_{\alpha \in \mathbb{Z}^m} c_\alpha u^\alpha$. Therefore, since μ is commutator bounded, sufficient condition for a to be in $\mathcal{A}_{\sigma,\mu}^{(n)}$ is that the sequence $c_\alpha |\alpha|^n \in \mathcal{L}^1(\mathbb{Z}^m)$. Subsequently, the sufficient condition for $a \in \mathcal{A}_{\sigma,\mu}^{(\infty)}$ is that this condition holds for all $n \in \mathbb{N}$. Observe that the set of all such elements is dense in A_σ.

It should also be noted that by the construction of the algebra $\mathcal{A}_\sigma$ the elements a satisfying the property $c_\alpha|\alpha|^n \in \mathcal{L}^1(\mathbb{Z}^m)$ form an algebra. Indeed, let a_1, a_2 be two such elements. Then for the product a_1a_2 we have by definition

$$a_1a_2 = \sum_\alpha \Big(\sum_{\beta+\gamma=\alpha} c_{1,\beta}c_{2,\gamma}e^{2\pi i\sigma(\beta,\gamma)} \Big) u^\alpha$$

where $c_{1,\beta}$ and $c_{c,\gamma}$ are the coefficients for a_1 and a_2 respectively, and

$$\sigma(\beta,\gamma) = \sum_{j=1}^{m}\sum_{k=1}^{m-j} \beta_k\gamma_j\sigma_{kj}$$

Thus, if c_α is the α's coefficient of $a := a_1a_2$, then

$$|c_\alpha| \leq \sum_{\beta+\gamma=\alpha} |c_{1,\beta}||c_{2_\gamma}|$$

The latter number is, in turn, the α's entry of the convolution product of the sequences $\{|c_{1,\alpha}|\}$ and $\{|c_{2,\alpha}|\}$. If both these sequences satisfy the condition $|c_{i,\alpha}||\alpha|^n \in \mathcal{L}^n(\mathbb{Z}^m)$, then so does their convolution product, hence we also have that the sequence $\{c_\alpha|\alpha|^n \in \mathcal{L}(\mathbb{Z}^m)\}$.

It should be noted that the sufficient condition for a to be in $\mathcal{A}_{\sigma,\mu}^{(\infty)}$ mentioned here is much more restrictive than the one used in noncommutative geometry. We recall that the algebra of smooth functions on noncommutative tori used in the definition of spectral triples on them is declared to be algebra of Schwatrz functions (also known as rapidly decreasing functions), i.e. the algebra $\mathcal{S}(A_\sigma)$ of elements $a \in A_\sigma$ satisfying the condition

$$\sup_{\alpha\in\mathbb{Z}^m} (1 + \|\alpha\|^2)^n |c_\alpha|^2 < \infty$$

The algebra $\mathcal{S}(A_\sigma)$ was constructed in such a way that it would be an analogue of smooth sunctions on the commutative torus (in fact, $C^\infty(\mathbb{T}^2) \supseteq \mathcal{S}(\mathbb{T}^2)$). So it seems very plausible that for any commutator bounded analytic fréchetization μ there is a system of operator algebras $\mathcal{A}_{\sigma,\mathcal{S}}^{(n)}$ such that

- $\mathscr{A}_{\sigma,\mathcal{S}} := \{\mathcal{A}_{\sigma,\mathcal{S}}^{(n)}\}$ is an ∞-smooth system,
- $\mathcal{A}_{\sigma,\mathcal{S}}^{(n)} \supseteq \{a \in A_\sigma \mid \sup_{\alpha\in\mathbb{Z}^m}(1+|\alpha|^2)^n|c_\alpha|^2 < \infty\}$ as a pre-C^*-algebra.
- $\Psi_\mu^{(\infty)}(\mathscr{A}_{\sigma,\mathcal{S}}, B) \to \mathrm{KK}(\mathcal{A}_\sigma, B)$ for any C^*-algebra B.

The author does not exclude the possibility that the smooth system $\mathscr{A}_\sigma$ we have constructed above satisfies these conditions if we take a sufficiently large set Λ of isomorphism classes of C^*-algebras in the construction. However, proving this or the contrary will need further development.

2.2 Product of Unbounded KK-cycles

2.2.1 Smooth Modules

The notion of smooth modules we are defining in this subsection generalizes the notion of smooth modules that appear in [28].

Definition 2.2.1. Let A be a C^*-algebra and let $\mathscr{A}$ be a smooth system on it. We say that a Hilbert C^*-B-module E has an *n-smooth structure* (or is *n-smooth*) *with respect to* $\mathscr{A}$ if there is an approximate unit $u_\alpha := \sum_{j=1}^{k_\alpha} x_j^\alpha \otimes x_j^\alpha$ in $\mathbb{K}_A(E)$, such that $\{u_\alpha\}$ defines a structure of stuffed module on E with respect the algebra $\mathcal{A}^{(k)}$ for all $k \leq n$. If $\mathrm{ord}\mathscr{A} = \infty$ and the property holds for all $n \in \mathbb{N}$, the module will be called ∞-*smooth*. We shall denote

$$\mathcal{E}^{(n)} := \{\xi \in E \mid \langle x_j^\alpha, \quad \sup_k \| \sum_{i=1}^{k_\alpha} e_i \langle x_j^\alpha, \xi\rangle \|_{\mathcal{A}^{(n)}} < \infty\}$$

and call it the *n-smooth submodule of* E (with respect to $\mathscr{A}$, $\{u_\alpha\}$).

Proposition 2.2.2. *Let* $\mathscr{A}$ *be a smooth system on* A*,* E *be a Hilbert* C^**-module over* A *and* $\{u_\alpha\}$ *define a smooth system* E *(of any order). Then*

- *The inclusion map* $\mathcal{E}^{(n+1)} \to \mathcal{E}^{(n)}$ *is completely bounded.*
- *If* $\mathscr{A}_1$ *is another smooth system on* A*, such that the same approximate unit* $\{u_\alpha\}$ *defines a smooth system* $\{\mathcal{E}_1^{(n)}\}$ *on* E *with respect to* $\mathscr{A}_1$*, and there is a completely bounded inclusion* $\mathcal{A}_1^{(n)} \to \mathcal{A}^{(n)}$ *for all* $n \leq \mathrm{ord}(\mathscr{A}_1)$*, then there are completely bounded inclusions* $\mathcal{E}_1^{(n)} \to \mathcal{E}^{(n)}$*.*

Proof. Follows immediately from Proposition 1.2.52. □

By the properties of Haagerup tensor product we have that there are morphisms $\mathrm{CB}^*_{\mathcal{A}^{(n+1)}}(\mathcal{E}^{(n+1)}) \to \mathrm{CB}^*_{\mathcal{A}^{(n)}}(\mathcal{E}^{(n)})$ and $\mathrm{CB}^*_{\mathcal{A}_1^{(n)}}(\mathcal{E}_1^{(n)}) \to \mathrm{CB}^*_{\mathcal{A}^{(n)}}(\mathcal{E}^{(n)})$, given simply by setting $T \mapsto T \otimes \mathrm{Id}_{\mathcal{A}^{(n)}}$. It also follows immediately form Proposition 1.2.58, that if D is an unbounded regular operator on $\mathcal{E}^{(n+1)}$ or $\mathcal{E}_1^{(n)}$, then the operator $D \otimes \mathrm{Id}_{\mathcal{A}^{(n)}}$ is an unbounded regular operator on $\mathcal{A}^{(n)}$. From the Lemma 1.2.66 we also obtain an analogous result for the connections.

We are now in the position to give an example, clarifying why we stick to the notion of almost rigged modules rather that just rigged modules.

Example 2.2.3. Let $\mathcal{A}_\theta$ be the C^*-algebra of the noncommutative 2-torus. It is a well-known fact that the algebra $\mathcal{A}_\theta$ contains nontrivial projections, called Powers-Rieffel projections. Let B be another C^*-algebra and (Y, D) be an unbounded (A, B)-KK-cycle. We may, for example, take $B = \mathbb{C}$ and (Y, D) be the Hilbert space and unbounded operator which are used in the definition of real spectral triple on the noncommutative torus (see, for example

[23], [34]). Denote $\mathcal{A}_\theta^{(n)} := A_\theta \cap \mathrm{Sob}_{mes,D}^{(n)}$, with the operator structure induced by the one on $\mathrm{Sob}_{mes,D}^{(n)}$, and set $\mathscr{A}_\theta := \mathcal{A}_\theta^{(n)}$.

It is a well-known fact that the algebra $\mathcal{A}_\theta$ contains nontrivial projections, called Powers-Rieffel projections. Since $\mathcal{A}_\theta^{(n)}$ are by the construction pre-C^*-subalgebras of A_θ, there is a Powers-Rieffel projection $p \in \mathcal{A}_\theta^{(\infty)} \subseteq \mathcal{A}_\theta^{(n)}$. Consider a projector of the form $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \in M_2(\mathcal{A}^{(\infty)})$. Set $E = p(\mathcal{A}_\theta)^2$. The approximate unit

$$u_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ p \end{pmatrix} \otimes \begin{pmatrix} 0 \\ p \end{pmatrix}$$

Let $\mathcal{E}^{(n)}$ be the corresponding smooth submodules. But now if we want $\mathcal{E}^{(n)}$ to be *rigged* over $\mathcal{A}_\theta^{(n)}$, we need to have $\|p\|_n = 1$ for all $n \in \mathbb{N}$. This will obviously *not* hold for an arbitrary $p \in \mathcal{A}^{(\infty)}$.

Moreover, there arises the following condition, that looks very artificial. By the definition of Mesland fréchetization the maps $\mathcal{A}_\theta^{(n+1)} \to \mathcal{A}_\theta^{(n)}$ are completely contractive, so that $\mathcal{E}^{(n)}$ will be rigged whenever $\mathcal{E}^{(n+1)}$ is rigged. However, if we encounter such N that $\|p\|_N > 1$, we would not be able to work with the modules $\mathcal{A}^{(n)}$ for $n \geq N$. The existence of such N is most probably the case for the smooth system coming out of the unbounded KK-cycle defining the spectral triple on the noncommutative torus that we have mentioned above.

All these problems are avoided when we work with almost rigged modules and completely bounded inclusions.

2.2.2 Transverse Unbounded Operators, Second Fréchetization

The notion of transversality of unbounded regular operators was given in [28] in the context of two unbounded KK-cycles, and is one of the key tools in the construction of the Kasparov product of these cycles. In this subsection we generalize this notion, imposing the condition of transversality of unbounded KK-cycle with respect to a smooth system on a C^*-algebra.

Definition 2.2.4. Let A and B be C^*-algebras and (E, T) be an unbounded (A, B)-KK-cycle. Let $\mathscr{B}$ be a smooth system on a C^*-algebra B and an approximate unit $\{u_\alpha\}$ in $\mathbb{K}_B(E)$ define a smooth structure on F with respect to $\mathscr{B}$. Then a *second fréchetization* is a map $\mu\colon (A, \mathscr{B}, F, \{u_\alpha\}, T, n) \mapsto \mathcal{A}_{\mu,T}^{(n)}$, where $\mathcal{A}_{\mu,T}^{(n)}$ is an operator algebra isomorphic to a subalgebra of A. The operator T would be called *C^n-transverse (with respect to fréchetization μ)* if system $\{\mathcal{A}_{\mu,T}^{(k)}\}_{k=0}^n$ satisfies the conditions of a smooth system from Definition 2.1.1.

Now, this definition is again way too abstract. We give it because it may become useful in future. Let us descend to a more practical approaches, that is related to the discussion we've had in the beginning of the Subsection 2.1.4. By μ we shall mean either commutator of Mesland fréchetizaton.

First of all, let E be a Hilbert C^*-module over B with an n-smooth structure with respect to $\mathscr{B}$. We denote the corresponding smooth modules by $\mathcal{E}^{(0,n)}$. Let now T be an unbounded regular operator on $\mathcal{E}^{(0,n)}$. We denote $\mathcal{E}^{(1,n)} := \mathfrak{G}(T)$. It is for now not known in general, whether T induces an analogue of the Sobolev chain of modules over $\mathcal{E}^{(0,n)}$, although there are some particular cases when such a property holds and we hypothesize that it could be done for every unbounded regular operator on a stuffed module. We are going to present such kind of operators in the examples to next subsections. Suppose that T satisfies this property, there is a Sobolev chain

$$\cdots \subseteq \mathcal{E}_T^{(j,n)} \subseteq \mathcal{E}_T^{(j-1,n)} \subseteq \cdots \subseteq \mathcal{E}^{(0,n)}$$

on $\mathcal{E}^{(0,n)}$. Then analogously to the case when there were no smooth structure on B, we may introduce the operators of the form $\pi_T^k(\cdot)$, $\theta_T^k(\cdot)$ or $\Theta_T^k(\cdot)$, but now as operators acting on $\mathcal{E}^{(0,n)}$, and thus define the algebras $\mathrm{Sob}^{(k,n)}$. If finally (E,T) was an unbounded (A,B)-KK-cycle, one may define $\mathcal{A}_{\mu,T}^{(k,n)} = \mathrm{Sob}_{\mu,T}^{(k,n)} \cap A$ with the norm induced by the one on $\mathrm{Sob}_T^{(k,n)}$. Now, T is called *C^n-transverse to the smooth structure $(E,\{u_j\})$ with respect to fréchetization μ* if $\{\mathcal{A}_{\mu,T}^{(k,k)}\}_{k=0}^n$ forms a smooth system. The definition of C^∞-transversality is given analogously.

Finally, we indicate the corollary of Proposition 1.2.60.

Proposition 2.2.5. *Let $\mathscr{B}$ and $\mathscr{B}_1$ be two smooth systems on B, such that there is a cb-inclusion $\mathcal{B}^{(n)} \to \mathcal{B}_1^{(n)}$ for all $n \leq N \leq \max\{\mathrm{ord}\mathscr{B}, \mathrm{ord}\mathscr{B}_1\}$. Let $\mathcal{E}^{(0,n)}$, $\mathcal{E}_1^{(0,n)}$ be stuffed modules, defined by the same approximate unit $\{u_\alpha\}$ on a Hilbert C^*-B-module E. Let (E,T) be an unbounded (A,B)-*KK*-cycle, such that T is N-transverse smooth on E with respect to $\mathscr{B}$ and $\{u_\alpha\}$. Then T is transverse smooth on E with respect to $\mathscr{B}_1$, $\{u_\alpha\}$.*

Proof. Since $\mathcal{E}_1^{(n)} \approxeq \mathcal{E}^{(n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \mathcal{B}_1^{(n)}$, the proof follows straightforward from Proposition 1.2.60. We only need to observe, that since $\mathcal{E}_T^{(j,n)} \to \mathcal{E}_{1,T}^{(j,n)} = \mathcal{E}^{(j,n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \mathcal{B}_1^{(n)}$ is a *cb*-inclusion given by, we have that there is a CB-inclusion $\mathrm{Sob}_1^{(j,n)} \to \mathrm{Sob}_{1,T}^{(j,n)}$. Therefore $A \cap \mathrm{Sob}_{1,T}^{(j,n)}$ is dense in A, and so T is indeed transverse smooth. □

Remark 2.2.6. Originally, the transversality was given in terms of two unbounded operators. In our notation, the operators D and T will be called C^n-transverse if T is C^n-transverse to the smooth structure $(E,\{u_j\})$ over the smooth system $\{\mathcal{B}_{\mu,D}^{(n)}\}$.

2.2.3 Transverse Smooth Connections

Definition 2.2.7. Let B be a C^*-algebra with a smooth system $\mathscr{B}$, E be a Hilbert C^*-module over B with some smooth structure with respect to $\mathscr{B}$ and T be an unbounded operator on E, which is C^n-transverse smooth on E with respect to $\mathscr{B}$. Finally let ∇ be a connection on $\mathcal{E}^{(0,n)}$. Then ∇ is said to be C^n-transverse if the operators

$$\mathrm{ad}_T^n(\nabla)(T \pm i)^{-n+1}, \quad \text{and} \quad (T \pm i)^{-n+1}\mathrm{ad}_T^n(\nabla)$$

extend to completely bounded operators $\mathcal{E}^{(0,n)} \to \mathcal{E}^{(0,n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)})$.

In case when the smooth system on a C^*-algebra B is trivial, we obtain a definition of *n-smooth connection* of [28].

Now let $\mathscr{B}_1$ be another smooth system on B, such that there is a completely bounded inclusion $\mathcal{B}^{(k)} \hookrightarrow \mathcal{B}_1^{(k)}$ for all $0 \leq k \leq n$. Then, as we have discussed above, there is a completely contractive inclusion $\mathcal{E}^{(0,n)} \hookrightarrow \mathcal{E}_1^{(0,n)}$, and, as we have seen earlier, the unbounded regular operators on $\mathcal{E}^{(0,n)}$ restrict to the ones on $\mathcal{E}_1^{(0,n)}$ and the same is true for the connections. We would like to show that every element of $\mathrm{CB}^*_{\mathcal{B}^{(n)}}(\mathcal{E}^{(0,n)}, \mathcal{E}^{(0,n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)}))$ extends uniquely to an element in $X \in \mathrm{CB}^*_{\mathcal{B}_1^{(n)}}(\mathcal{E}_1^{(0,n)}, \mathcal{E}_1^{(0,n)} \tilde{\otimes}_{\mathcal{B}_1^{(n)}} \Omega^1(\mathcal{B}_1^{(n)}))$. Indeed, by definition $\mathcal{E}^{(0,n)} \oplus \mathcal{H}_{\mathcal{B}^{(n)}} \approxeq \mathcal{H}_{\mathcal{B}^{(n)}}$ completely boundedly, and the same holds for $\mathcal{E}_1^{(0,n)}$. Therefore we may work with the maps $\mathcal{H}_{\mathcal{B}^{(n)}} \to \mathcal{H}_{\mathcal{B}^{(n)}} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)})$ and $\mathcal{H}_{\mathcal{B}_1^{(n)}} \to \mathcal{H}_{\mathcal{B}_1^{(n)}} \tilde{\otimes}_{\mathcal{B}_1^{(n)}} \Omega^1(\mathcal{B}_1^{(n)})$. But now, as it was observed earlier,

$$\mathcal{H}_{\mathcal{B}^{(n)}} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)}) \approxeq \mathcal{H} \tilde{\otimes} \Omega^1(\mathcal{B}^{(n)})$$

Now we observe that $\Omega^1(\mathcal{B}_1^{(n)}) = \mathcal{B}_1^{(n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)}) \tilde{\otimes}_{\mathcal{B}^{(n)}} \mathcal{B}_1^{(n)}$ so that

$$\mathcal{H} \tilde{\otimes} \Omega^1(\mathcal{B}_1^{(n)}) \approxeq \mathcal{B}_1^{(n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} (\mathcal{H} \tilde{\otimes} \Omega^1(\mathcal{B}^{(n)})) \tilde{\otimes}_{\mathcal{B}^{(n)}} \mathcal{B}_1^{(n)}$$

Therefore, we have that every element of $\mathrm{CB}^*_{\mathcal{B}^{(n)}}(\mathcal{H}_{\mathcal{B}^{(n)}}, \mathcal{H}_{\mathcal{B}^{(n)}} \tilde{\otimes}_{\mathcal{B}^{(n)}|} \Omega^1(\mathcal{B}^{(n)}))$ extends to an element in $\mathrm{CB}^*_{\mathcal{B}_1^{(n)}}(\mathcal{H}_{\mathcal{B}_1^{(n)}}, \mathcal{H}_{\mathcal{B}_1^{(n)}} \tilde{\otimes}_{\mathcal{B}_1^{(n)}} \Omega^1(\mathcal{B}_1^{(n)}))$, and since $\mathcal{B}^{(n)}$ is dense in $\mathcal{B}_1^{(n)}$ this extension is unique. Because of the stability property of $\mathcal{E}^{(0,n)}$ and $\mathcal{E}_1^{(0,n)}$, this picture may be restricted to the original case.

The last conclusion tells us that given an unbounded regular operator T on and smooth connection ∇ on $\mathcal{E}$, which are C^n-transverse smooth with respect to the smooth structure induced on E by the smooth system $\{\mathcal{B}^{(n)}\}$ and an approximate unit $\{u_j\}$, then these maps extend to an unbounded regular operator and connections satisfying the same properties with respect to $\{\mathcal{B}_1^{(n)}\}$. Indeed, as we have just seen, if the operators $\mathrm{ad}_T^n(\nabla)(T \pm i)^{-n+1}$ and $(T \pm i)^{-n+1}\mathrm{ad}_T^n(\nabla)$ belong to $\mathrm{CB}^*(\mathcal{E}^{(0,n)}, \mathcal{E}^{(0,n)} \tilde{\otimes} \Omega^1(\mathcal{B}^{(n)}))$, then they will extend to the elements of $\mathrm{CB}^*_{\mathcal{B}^{(n)}}(\mathcal{E}^{(0,n)}, \mathcal{E}^{(0,n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)}))$. Denote this extensions by a and b for short. We only need to observe, that if we take the extensions of T and ∇ to $\mathcal{E}_1^{(0,n)}$, that is, the operators $T\colon \mathcal{E}_1^{(0,n)} \supseteq \mathrm{Dom}(T) \to \mathcal{E}_1^{(0,n)}$ and $\nabla\colon \mathcal{E}_1^{(0,n)} \to \mathcal{E}_1^{(0,n)} \tilde{\otimes}_{\mathcal{B}_1^{(n)}} \Omega^1(\mathcal{B}_1^{(n)})$ then

$$\mathrm{ad}_T^n(\nabla)(T \pm i)^{-n+1}\colon \mathcal{E}_1^{(0,n)} \to \mathcal{E}_1^{(0,n)} \tilde{\otimes}_{\mathcal{B}_1^{(n)}} \Omega^1(\mathcal{B}_1^{(n)})$$

and

$$(T \pm i)^{-n+1}\mathrm{ad}_T^n(\nabla)\colon \mathcal{E}_1^{(0,n)} \to \mathcal{E}_1^{(0,n)} \tilde{\otimes}_{\mathcal{B}_1^{(n)}} \Omega^1(\mathcal{B}_1^{(n)})$$

will coincide with a and b respectively on a dense subspace of $\mathcal{E}_1^{(0,n)}$. Since a and b are completely bounded, the latter operators also extend to completely bounded, and so we prove the claim.

2.2.4 Product of Unbounded KK-Cycles: Theorem of Mesland and its Generalization

We have finally reached the point where we can establish the connection of all the theory defined before with the construction of an analogue of the Kasparov product for unbounded KK-cycles. In this subsection we are going to consider only the Mesland fréchetization, since it gives important results for interesting unbounded operators.

The main result of [28] reads

Theorem 2.2.8. *[conf. [28][Thm 6.2.3]] Let A, B and C be C^*-algebras and $n \in \mathbb{N}$. Let (E,T) and (F,D) be unbounded (A,B)- and (B,C)-KK-cycles. Suppose F is endowed with a n-smooth structure with respect to the smooth system $\mathscr{B}_{mes,D} := \{\mathcal{B}^{(k)}_{mes,D}\}_{k=1}^n$. Suppose also that T is n-transverse smooth with respect to the smooth system $\mathscr{B}_{mes,D}$, and ∇ is $n+1$-transverse smooth with respect to T and $\mathscr{B}_{mes,D}$. Then we have that the data*

$$(E \tilde{\otimes}_B F; T \tilde{\otimes} 1 + 1 \tilde{\otimes}_\nabla D)$$

where

$$(T \tilde{\otimes} 1 + 1 \tilde{\otimes}_\nabla D)(\xi \otimes \eta) := T\xi \otimes \eta + \nabla_D(\xi)\eta + \xi \otimes D\eta$$

is a regular (unbounded) selfadjoint operator on $F \tilde{\otimes}_B E$, and

$$[(E \tilde{\otimes}_B F; \mathfrak{b}(T \tilde{\otimes} 1 + 1 \tilde{\otimes}_\nabla D))] = [(E, \mathfrak{b}(T))] \times [(F, \mathfrak{b}(D))]$$

as elements of $\mathrm{KK}_0(A,C)$, $\mathrm{KK}_0(A,B)$ and $\mathrm{KK}_0(B,C)$, with $\times$ being the Kasparov product.

Having this result, we may formulate the following corollary.

Theorem 2.2.9. *Let B be a C^*-algebra with a smooth system $\mathscr{B}$ on it. Let (E,T) be an unbounded even (A,B)-KK-cycle and there is an approximate unitary on $\{u_\alpha\} \in \mathbb{K}_B(E)$ defining a $\mathscr{B}$-smooth structure on E, with T selfadjoint and being (at least) C^n-transverse with respect to this smooth system. Suppose also that there is a Hermitian connection ∇ on E which is at least C^{n+1}-transverse smooth with respect to the smooth system on E and T. Then for any C^*-algebra C and any element $(Y,D) \in \Psi_0^{(n)}(\mathscr{B},C)$ the data $(E \tilde{\otimes}_B Y, T \otimes 1 + 1 \otimes_\nabla D)$ forms a n-smooth unbounded (A,C)-KK-cycle. Moreover, $[(E \tilde{\otimes}_B Y, \mathfrak{b}(T \otimes 1 + 1 \otimes_\nabla D))] = [(E, \mathfrak{b}(T))] \times [Y, \mathfrak{b}(D)]$ as elements of $\mathrm{KK}(A,C)$, $\mathrm{KK}(A,B)$ and $\mathrm{KK}(B,C)$ respectively.*

Proof. We recall that by definition $(Y,D) \in \Psi^{(n)}_{mes}(\mathscr{B},C)$ means that there is a *cb*-inclusion $\mathcal{B}^{(k)} \to \mathcal{B}^{(k)}_{mes,D}$ for all $k \leq n$. Therefore, if (E,T) satisfies all the above conditions, then,

- The approximate unit $\{u_\alpha\}$ defines a structure of rigged module $\mathcal{E}^{(0,k)}_{mes,D}$ on E with respect to the system $\mathscr{B}_{mes,D}$, and $\mathcal{E}^{(0,k)}_{mes,D} \approxeq \mathcal{E}^{(n)} \tilde{\otimes}_{\mathcal{B}^{(k)}} \mathcal{B}^{(k)}_{mes,D}$ for all $k \leq n$.

- By 2.2.5 since T is n-transverse smooth on $\mathcal{E}^{(n)}$, it will be n-transverse smooth on $\mathcal{E}^{(0,n)}_{mes,D}$.
- By the discussion in the subsection 2.2.3, the connection ∇ on $\mathcal{E}^{(n)}$ will be $(n+1)$-transverse smooth on $\mathcal{E}^{(0,n)}_{mes,D}$

Thus, we see that the moudle $\mathcal{E}^{(0,n)}_{mes,D}$, the operator T and the connection ∇ satisfy the conditions of the Theorem 2.2.8, and so the cycle $(E \tilde{\otimes}_B Y, T \otimes 1 + 1 \otimes_\nabla D)$ is a well defined unbounded (A, C)-KK-cycle, which is n-smooth and satisfies the conditions of compatibility with the KK-groups. □

We illustrate this result with an example.

Example 2.2.10. Let $\theta \in [0, 2\pi)$ be a real number and consider a noncommutative 2-torus algebra A_θ. We take a smooth system $\mathscr{A}_\theta := \mathscr{A}_{\theta,mes}$ to be the smooth system on A_θ as we have defined above in the example devoted to the smooth systems for noncommuative tori. It is a well-known fact that the algebra of the noncommutative 2-torus contains a nontrivial projector, also called the Powers-Rieffel projector. Since the algebra $\mathcal{A}^{(\infty)}_\theta$ is by construction a pre-C*-algebra, there exists such Powers-Rieffel projector that belongs to $\mathcal{A}^{(\infty)}_\theta$. We denote this projector by q. Now, choose $\kappa_1, \kappa_2 \in \mathbb{N}$ such that κ_1 and κ_2 are mutually prime. We form a projector

$$q_{\kappa_1,\kappa_2} = \mathrm{diag}\{\underbrace{1,\ldots,1}_{\kappa_1}, \underbrace{q,\ldots,q}_{\kappa_2}\}$$

on $A^{\kappa_1+\kappa_2}_\theta$. We denote

$$E := (\ell^2(\mathbb{Z}) \tilde{\otimes} q_{\kappa_1,\kappa_2} A^{\kappa_1+\kappa_2}_\theta) \oplus (\ell^2(\mathbb{Z}) \tilde{\otimes} q_{\kappa_1,\kappa_2} A^{\kappa_1+\kappa_2}_\theta)$$

For simplicity denote the elements of $\ell^2(\mathbb{Z}) \tilde{\otimes} q_{\kappa_1,\kappa_2} A^{\kappa_1+\kappa_2}_\theta$ by $\sum_{j=-\infty}^{+\infty} w^j \xi_j$ where $\xi_j \in q_{\kappa_1,\kappa_2} A^{\kappa_1+\kappa_2}_\theta$ and $j \in \mathbb{Z}$. According to [15] we have that

$$\mathbb{K}_{A_\theta}(q_{\kappa_1,\kappa_2} A^{\kappa_1+\kappa_2}_\theta) = A_\nu$$

where

$$\nu = \frac{\kappa_1 + \kappa_2\theta}{a + b\theta}$$

and $\left(\begin{smallmatrix} \kappa_1 & \kappa_2 \\ a & b \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Define an antisymmetric matrix

$$\sigma = \begin{pmatrix} 0 & \nu & \alpha \\ -\nu & 0 & \beta \\ -\alpha & -\beta & 0 \end{pmatrix}$$

with α and β being arbitrary positive real numbers (from the semi-interval $[0; 2\pi)$ for uniformity) and form a noncommutative 3-torus A_σ using this matrix. We denote the generators of A_σ by u, v and w. Then, according to the previous observation, we may form an action of A_σ on E by setting

$$u^x v^y w^z \cdot w^j \xi_j = \exp((\alpha x + \beta y)(z+j)) w^{z+j}(u^x v^y \xi_j)$$

on each copy of $(\ell^2(\mathbb{Z}) \tilde{\otimes} q_{\kappa_1,\kappa_2} A_\theta^{\kappa_1+\kappa_2})$ and then extending this action by linearity.

Now we define a derivation ∂ on $\ell^2(\mathbb{Z}) \tilde{\otimes} q_{\kappa_1,\kappa_2} A_\theta^{\kappa_1+\kappa_2}$ by setting

$$\partial(w^j \xi_j) = j w^j \xi_j$$

construct an unbounded operator T on E by setting

$$T = \begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix}$$

This operator is obviously selfadjoint densely defined, and the compactness of its resolvent may be checked directly.

Finally, we would like to form a connection on E. First of all, there is a natural universal Hermitian connection on A_θ given by $a \mapsto 1 \otimes da$, which we may extend to $A_\theta^{\kappa_1+\kappa_2}$. Denote it by $\tilde{\nabla}'$. We define a connection on $q_{\kappa_1,\kappa_2} A_\theta^{\kappa_1+\kappa_2}$ by setting $\tilde{\nabla} = q_{\kappa_1,\kappa_2} \nabla$. The fact that this is a Hermitian connection follows from [18]. Finally, we set a connection ∇ on E by setting

$$\nabla(w^j \xi_j) = w_j \tilde{\nabla} \xi_j$$

on each copy of $\ell^2(\mathbb{Z}) \tilde{\otimes} q_{\kappa_1,\kappa_2} A_\theta^{\kappa_1+\kappa_2}$.

By the construction T maps the space

$$\text{span}_{j,t\in\mathbb{Z}} \{ \left(\begin{smallmatrix} w^j \xi_j \\ 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 \\ w^t \xi_t \end{smallmatrix} \right) \}$$

where

$$\xi_j, \xi_t \in q_{\kappa_1,\kappa_2} (\mathcal{A}_\theta^{(\infty)})^{\kappa_1+\kappa_2}$$

onto itself. Therefore T may be restricted to an unbounded regular operator $\mathcal{E}^{(0,n)}$ for all $n \in \mathbb{N} \cup \{0\}$. Moreover, by the same observation it follows that T defines a Sobolev chain on each $\mathcal{E}^{(0,n)}$. The spaces $\mathcal{E}^{(k,n)}$ in this case consist of the elements of the form

$$\sum_{j\in\mathbb{Z}} \begin{pmatrix} w^j \xi_j \\ 0 \end{pmatrix} + \sum_{t\in\mathbb{Z}} \begin{pmatrix} 0 \\ w^t \xi_t \end{pmatrix}$$

such that $\sum_{j\in\mathbb{Z}} |j|^k \|\xi_j\|_k < \infty$ and $\sum_{t\in\mathbb{Z}} |t|^k \|\xi_t\|_k < \infty$.

the universal connection ∇ obviously restricts to $\mathcal{E}^{(n,n)}$. Moreover, we have that

$$\begin{aligned}
[T, \nabla] \begin{pmatrix} w_j \xi_j \\ 0 \end{pmatrix} &= \left(\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} + \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} \begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix} \right) \begin{pmatrix} w_j \xi_j \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix} \begin{pmatrix} w_j \tilde{\nabla} \xi_j \\ 0 \end{pmatrix} + \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} \begin{pmatrix} 0 \\ j w_j \xi_j \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -j w_j \tilde{\nabla} \xi_j \end{pmatrix} + \begin{pmatrix} 0 \\ j w_j \tilde{\nabla} \xi_j \end{pmatrix} \\
&= 0
\end{aligned}$$

and we have an analogous picture for $\begin{pmatrix} 0 \\ w^t\xi_t \end{pmatrix}$.

Therefore ∇ is ∞-transverse smooth connection with respect to T and $\mathscr{A}_\theta$.

Finally, we observe that the elements of the algebra A of the form $u^x v^y w^z$ extend to bounded operators on $\mathcal{E}^{(n,n)}$. therefore we may apply the theorem of Mesland, and we have that for all $(Y, D) \in \Psi_0^{(n)}(\mathscr{A}_\theta, B)$ the operator $S = T \otimes 1 + 1 \otimes_\nabla D$ defines an n-smooth system $\mathscr{A}_{\sigma,S}$ on the C^*-algebra A_σ for $n \in \mathbb{N} \cup \{\infty\}$. Since the product of unbounded regular operators accords with the Kasparov product, the construction of this example actually generalizes the map given by

$$[(E, \mathfrak{b}(T))] \times \mathrm{KK}(A_\theta, \cdot) \to \mathrm{KK}(A_\sigma, \cdot)$$

for the smooth case. However, now we may say that

$$[(E, \mathfrak{b}(T))] \times [Y, \mathfrak{b}(D)] = [E \tilde{\otimes}_B Y, \mathfrak{b}(T \otimes 1 + 1 \otimes_\nabla D)]$$

for every $(Y, D) \in \Psi_{mes}^{(\infty)}(\mathscr{A}_\theta, \cdot)$. Since the set $\{[(Y, \mathfrak{b}(D))] \mid (Y, D) \in \Psi_{mes}^{(\infty)}(\mathscr{A}_\theta, C)\}$ contains all the elements of $\mathrm{KK}(A_\theta, C)$, we have presented the way to compute the concrete values of of Kasparov product with the particular element $[(E, \mathfrak{b}(T))]$, avoiding the complications of Kasparov technical lemma. Moreover, since the smoothness of the operators has been taken into consideration, we have preserved more properties than we could have preserved using the bounded picture. We suppose that these properties may serve for finding invariants coming the unbounded KK-cycles, which may be finer than the ones provided by classical KK-theory.

2.3 Prospects

2.3.1 An Approach to a Category of C^*-Algebras with Smooth Structures

One of the main directions for the further research arises from the question, whether we may form a category of C^*-algebras with smooth systems, and what kind of morphisms should this category have.

To be more concrete, we have seen that if T is, for instance, C^1-transverse smooth and ∇ is C^2-transverse smooth with respect to a smooth system $\mathscr{B}$ on B and an approximate unit $\{u_\alpha\}$ on a Hilbert C^*-B-module E, then $(E \tilde{\otimes}_B Y, T \otimes 1 + 1 \otimes_\nabla D)$ is an unbounded (A, C)-KK-cycle for all $(Y, D) \in \Psi_{mes}^{(1)}(\mathscr{B}, C)$. However, it is still not known, whether in general there exists a smooth system $\mathscr{A}$ on A such that $(E \tilde{\otimes}_B Y, T \otimes 1 + 1 \otimes_\nabla D) \in \Psi_{mes}^{(1)}(\mathscr{A}, C)$ for all $(Y, D) \in \Psi_{mes}^{(1)}(\mathscr{B}, C)$.

For now this picture is obtained in detail only for the case when E is a finitely generated free module over B. We present this case as an example:

Example 2.3.1. Let B be a C^*-algebra and let $\mathscr{B}$ be a smooth system on B. We shall assume that $\mathrm{ord}\mathscr{B} = \infty$. We also add the condition that the norms ${}_m\|\cdot\|_n$ are analytic with respect to ${}_m\|\cdot\|_{n-1}$ on the algebra $M_m(\mathcal{B}^{(n)})$ for all $n \in \mathbb{N}$ and all $m \in \mathbb{N}$. This condition does not seem to be too obligatory in practice, because it holds for smooth systems obtained by *mes* and ad fréchetizations and will also hold for smooth systems constructed in the Theorem 2.1.7 with $\mu \in \{mes, \mathrm{ad}\}$.

We set $E = B^m$. The module E may be endowed with a smooth structure with respect to $\mathscr{B}$ by setting $\mathcal{E}^{(n)} := (\mathcal{B}^{(n)})^m$. We take a $*$-subalgebra $\mathcal{A} \subseteq M_m(\mathcal{B}^{(\infty)})$, such that $\mathcal{A}$ contains the identity of $M_m(B)$ and its closure with respect to the C^*-norm on $M_m(\mathcal{B}^{(\infty)})$ is a C^*-algebra. Denote $\mathcal{A}^{(n)} := \overline{\mathcal{A}}$, with the closure taken with respect the norms ${}_m\|\cdot\|_n$ on $M_m(\mathcal{B}^{(n)})$ for $n = \{0\} \cup \mathbb{N}$. We make the algebras $\mathcal{A}^{(n)}$ inherit the operator algebra structure form the ones on $M_m(\mathcal{B}^{(n)})$. Since the maps $\mathcal{B}^{(n+1)} \to \mathcal{B}^{(n)}$ are completely bounded, so are the inclusion maps $M_m(\mathcal{B}^{(n+1)}) \to M_m(\mathcal{B}^{(n)})$, and therefore $\mathcal{A}^{(n+1)} \to \mathcal{A}^{(n)}$ are completely bounded inclusions. We also note that the norms ${}_m(\mathcal{B}^{(n+1)})$ are by the conditions analytic with respect to ${}_m(\mathcal{B}^{(n)})$, and so we have that the norm on $\mathcal{A}^{(n+1)}$ is analytic with respect to $\mathcal{A}^{(n)}$. Hence the algebras $\mathcal{A}^{(n)}$ are stable under the holomorphic functional calculus on A. Finally, since all $\mathcal{A}^{(n)}$ contain the unit of $M_m(\mathcal{B}^{(n)})$, the maps $\mathcal{A}^{(n+1)} \to \mathcal{A}^{(n)}$ as well as $\mathcal{A}^{(n)} \to M_m(B)$ are essential. We also have that $\mathcal{A}^{(\infty)}$ is dense in A and stable under holomorphic functional calculus on A.

Thus, we have shown that the system of nested subalgebras $\mathscr{A} := \{\mathcal{A}^{(n)}\}_{n=1}^{\infty}$ satisfies the conditions of a smooth system.

Now, since E is finitely generated over B, all regular operators on E are actually bounded. Let T be such an operator. Then for T to be n-transverse smooth with respect to the smooth system $\mathscr{B}$ we should just have $T \in \mathrm{CB}^*_{\mathcal{B}^{(n)}}(\mathcal{E}^{(n)})$ and be odd with respect to the grading on E. Therefore, if there is a connection ∇_1 on $\mathcal{E}^{(\infty)}$, then we shall automatically have that $\mathrm{ad}^n_T(\nabla_1)$ will extend to a completely bounded operator on $\mathcal{E}^{(n)}$ for all n.

We would like to show that for every C^*-algebra C and for any unbounded KK-cycle $(Y, D) \in \Psi^{(n)}_{mes}(\mathscr{B}, C)$ for $n \in \mathbb{N}$ we have that $(E \tilde{\otimes}_B Y, T \otimes 1 + 1 \otimes_{\nabla_1} D) \in \Psi^{(n)}_{mes}(\mathscr{A}, C)$.

Indeed, first of all, we have that $T\otimes_1 \in \mathrm{End}_C(E \tilde{\otimes}_B Y)$ by the construction. Second, we may present $\nabla_1 := \nabla + b$, where ∇ is the Grassmanian connection on E and $b\colon E \to E \tilde{\otimes}_B \Omega^1(B)$ is a completely bounded B-linear map. Therefore, we may regard

$$
\begin{aligned}
(T \otimes 1 + 1 \otimes_{\nabla_1} D)(\xi \otimes \eta) &= T\xi \otimes \eta + \nabla_1(\xi)\eta + \xi \otimes D_\eta \\
&= T\xi \otimes \eta + \nabla(\xi)\eta + b(\xi)\eta + \xi \otimes D\eta \\
&= (1 \otimes_\nabla D)\xi \otimes \eta + (T\xi \otimes \eta + b(\xi)\eta)
\end{aligned}
$$

Now, we need to verify that $1 \otimes_\nabla D \in \Psi^{(\infty)}(\mathscr{A}, C)$. But since $E = B^m$ we may write $\xi = (b_1, \dots, b_m)^\intercal$, and therefore

$$
(1 \otimes_\nabla D)(\xi \otimes \eta) = (1 \otimes [D; b_1], \dots, 1 \otimes [D; b_m])^\intercal \eta + (b_1, \dots, b_m)^\intercal \otimes D\eta = \sum_{j=1}^{m} e_j \otimes D b_j \eta
$$

where $\{e_j\}$ is the standard basis of $\mathcal{E}^{(n)} = (\mathcal{B}^{(n)})^m$. Thus, we actually obtain that $1 \otimes_\nabla D = \mathrm{diag}_m(D)$. As we have already seen in the part devoted to Mesland fréchetization,

${}_{mk}\|(b_{ij})\|_{mes,n,D} = {}_{k}\|(b_{ij})\|_{mes,n,\mathrm{diag}_m D}$ for all $k \in \mathbb{N}$ and all $n \in \mathbb{N}$. Thus, there is a completely isometric isomorphism

$$(M_m(B))^{(n)}_{mes,\mathrm{diag}_m D} \approxeq M_m(\mathcal{B}^{(n)}_{mes,D}) \tag{2.8}$$

Now, since $(Y, D) \in \Psi^{(\infty)}_{mes}(\mathcal{B}^{(n)}, C)$ we have by definition that there is a completely bounded map $\mathcal{B}^{(n)} \to \mathcal{B}^{(n)}_{mes,D}$, and, following the definition of completely bounded map means that the maps

$$M_k(\mathcal{B}^{(n)}) \to M_k(\mathcal{B}^{(n)}_{mes,D})$$

are uniformly bounded for all k. This in particular means that the maps

$$M_{mk}(\mathcal{B}^{(n)}) \to M_{mk}(\mathcal{B}^{(n)}_{mes,D})$$

are uniformly bounded. But since $\mathcal{A}^{(n)} \subseteq M_m(\mathcal{B}^{(n)})$, we have that the restriction of the above maps to $M_k(\mathcal{A}^{(n)})$ will also be uniformly bounded for all k, and this property holds for all $n \in \{0\} \cup \mathbb{N}$. Thus, the map $\mathcal{A}^{(n)} \to M_m(\mathcal{B}^{(n)})$ is completely bounded. Finally, we observe that the isomorphism 2.8 gives us a completely bounded isomorphism between the algebras $\mathcal{A}^{(n)}_{mes,\mathrm{diag}_m D}$ and $A \cap M_n(\mathcal{B}^{(n)}_{mes,D})$. Therefore $(E \otimes_B Y, 1 \otimes_\nabla D) \in \Psi^{(\infty)}_{mes}(\mathscr{A}, C)$.

Now, we just have to mark that the map

$$R\colon \xi \otimes \eta \mapsto T\xi \otimes \eta + b(\xi)\eta$$

is by construction a completely bounded (selfadjoint) operator on $\mathcal{E}^{(n)}$. Now, according to [28, Cor 4.8.5] we have that the smooth system $\mathcal{A}^{(n)}_{mes,\mathrm{diag}_m D+R}$ is equivalent to $\mathcal{A}^{(n)}_{mes,\mathrm{diag}_m D}$ whenever $R \in \mathrm{Sob}^{(n)}_{mes,\mathrm{diag}_m D}$. The latter condition will hold for all $T \in \mathcal{A}^{(n)}$ and all such B-linear operators b that restrict to a completely bounded operator of the form $b\colon \mathcal{E}^{(n)} = (\mathcal{B}^{(n)})^m \to (\mathcal{B}^{(n)})^m \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)})$.

Thus we have shown that, if $T \in \mathrm{CB}^*_{\mathcal{B}^{(n)}}\mathcal{E}^{(n)}$ and $\nabla_1 - \nabla \in \mathrm{CB}^*(\mathcal{E}^{(n)}, \mathcal{E}^{(n)} \tilde{\otimes}_{\mathcal{B}^{(n)}} \Omega^1(\mathcal{B}^{(n)}))$ for all $n \in \mathbb{N}$, then the data $(\mathscr{B}, \{u_\alpha\}, E, T, \nabla)$ define a morphism

$$\Psi^{(\infty)}_{mes}(\mathscr{B}, C) \to \Psi^{(\infty)}_{mes}(\mathscr{A}, C)$$

for any C^*-algebra C. Finally, if $\mathscr{A}_1$ is some other smooth system on A, such that $\mathcal{A}_1^{(n)} \hookrightarrow \mathcal{A}^{(n)}$ are completely bounded, then

$$\Psi^{(\infty)}_{mes}(\mathscr{B}, C) \to \Psi^{(\infty)}_{mes}(\mathscr{A}_1, C)$$

Thus, for $n \in \mathbb{N} \cup \{\infty\}$ we may construct a category, whose objects are C^*-algebras with n-smooth systems on them and the morphism are given by the data (E, T, ∇) as in the example.

An analogous picture should presumably hold for $E = pB^m$ where $p \in M_m(B)$ is a projector, such that in fact $p \in M_m(\mathcal{B}^{(\infty)})$. Unfortunately, even in this case we encounter additional complications, and for now the picture is more or less clear only for the first order of smoothness. We discuss this case in the next example:

Example 2.3.2. We take a C^*-algebra B and let $\mathcal{B} := \mathcal{B}^{(1)}$ be an operator $*$-subalgebra of B, such that the inclusion $\mathcal{B} \hookrightarrow B$ is completely bounded. We may regard $\mathscr{B} := \{\mathcal{B} \hookrightarrow B\}$ as a smooth system of order 1. We take $E := pB^m$, where p is a projector in $M_m(\mathcal{B})$. We denote $\mathcal{E} = p\mathcal{B}^m$. The space $\mathcal{E}$ becomes a $\mathcal{B}$-operator module, and the algebra $\mathrm{CB}^*_{\mathcal{B}}(\mathcal{E})$ inherits an operator algebra structure from $M_m(\mathcal{B})$ by identification $\mathrm{CB}^*_B(\mathcal{E}) = pM_n(\mathcal{B})p$.

We choose a complete subalgebra $\mathcal{A} \subseteq \mathrm{CB}^*_{\mathcal{B}}(\mathcal{E})$ with induced operator norm, and set A to be a completeion of $\mathcal{A}$ with respect to a C^*-norm on $\mathrm{End}^*_B(E)$. Since $\mathcal{B} \to B$ is a completely, so will, by definition, be the inclusion $\mathcal{A} \hookrightarrow A$, so that we may consider $\mathscr{A} := \{\mathcal{A}, A\}$ as smooth system of order 1.

As in the previous example, the regular B-linear operators on E are actually bounded, and so, in order for such an operator T to be 1-transverse with respect to $\mathscr{B}$, it suffices that $T \in M_m(\mathcal{B})$. The boundedness of $\mathrm{ad}^2_T(\nabla_1)$ follows immediately from the fact that T is bounded.

Now, let $(Y, D) \in \Psi^{(1)}_{mes}(\mathscr{B}, C)$ for some C^*-algebra C. For a Grassmanian connection ∇ on E we have that

$$\begin{aligned}(1 \otimes_\nabla D)(\xi \otimes \eta) &= \sum_{j=1}^m pe_j \otimes [D; b_j]\eta + \sum_{j=1}^m p(e_j b_j) \otimes D\eta \\ &= \sum_{j=1}^m pe_j \otimes ([D; b_j]\eta + b_j D\eta)\end{aligned}$$

where, again $\{e_j\}$ is the standard basis on $\mathcal{B}^m$ and the vectors $\xi = (b_1, \dots, b_m)^\intercal$ lay in $p\mathcal{B}^m \subseteq \mathcal{B}^m$. Therefore, since every element of $\mathcal{A} \subseteq pM_m(\mathcal{B})p$, we have that

$$[1 \otimes_\nabla D, a] = p[\mathrm{diag}_m D, a]$$

for $a \in A$. Since $A \subseteq pM_m(B)p$, so that $pa = a$, we have that

$$\pi^1_{1\otimes_\nabla D}(a) = \left(\begin{pmatrix} a & 0 \\ p[1 \otimes_\nabla D; a] & \gamma a \gamma \end{pmatrix}\right) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} a & 0 \\ [\mathrm{diag}_m D; a] & \gamma a \gamma \end{pmatrix}$$

From the previous example we already know, that ${}_m\|a\|_{mes,1,D} = {}_1\|a\|_{mes,1,\mathrm{diag}_m D}$. Therefore, by previous equation, we have that

$${}_m\|a\|_{mes,1,D} \geq {}_1\|a\|_{mes,1,1\otimes_\nabla D}$$

Thus, we obtain that there is a bounded map $\mathcal{A} \to \mathcal{A}^{(1)}_{mes,1,1\otimes_\nabla D}$ for all $D \in \Psi^{(1)}_{mes}(\mathscr{B}, C)$. Like in the previous example, we may prove that this map is also completely bounded, so that $(E \tilde{\otimes}_B Y, 1 \otimes_\nabla D) \in \Psi^{(1)}_{mes}(\mathscr{A}, C)$.

And, again, as in the previous example, we then may say, we may now say that $(E \tilde{\otimes}_B Y, T \otimes 1 + 1 \otimes_{\nabla'} D) \in \Psi^{(1)}_{mes}(\mathscr{A}, C)$ for some other Hermitean connection on $\mathcal{E}$ and selfadjoint $\mathcal{B}$-linear operator T on $\mathcal{E}$.

Of course, the result provided in the example encourages us to construct a category, whose object would be 1-smooth systems and the morphisms would be given by the data of the form (E, T, ∇) as above. The difficulty is that by theorem of Mesland, if there are unbounded (A_{i-1}, A_i)-KK-cycles (E_i, D_i), $i = 1, 2, 3$, with smooth connections ∇_1 and ∇_2 on E_1 and E_2, then we have that there is a canonical unitary equivalence between the unbounded (A_0, A_3)-KK-cycles

$$\begin{aligned}&(E_1 \tilde{\otimes}_{A_1} E_2 \tilde{\otimes}_{A_2} E_2; (D_1 \otimes 1_{E_2} + 1_{E_1} \otimes_{\nabla_1} D_2) \otimes 1_{E_3} + 1_{E_1 \tilde{\otimes}_{A_1} E_2} \otimes_{\nabla_2} D_3)\\ \sim_u &(E_1 \tilde{\otimes}_{A_1} E_2 \tilde{\otimes}_{A_2} E_2; D_1 \otimes 1_{E_2 \tilde{\otimes}_{A_2} E_3} + 1_{E_1} \otimes_{\nabla_1} (D_2 \otimes 1_{E_3} + 1_{E_2} \otimes_{\nabla_2} D_3))\end{aligned}$$

Therefore, although the data $(\mathcal{E}, T, \nabla)$ as in 2.3.2 induces a set-theoretical map $\Psi^{(1)}_{mes}(\mathscr{B}, \cdot) \to \Psi^{(1)}_{mes}(\mathscr{A}, \cdot)$, such maps do not in general satisfy the associativity property. So, even if we construct such a category as we described above, the construction of a good functor to the category of sets will require some additional considerations,

Replacing unbounded operators by their unitary equivalence class will not solve this problem, because then we obtain an ambiguity, which we show in the following example.

Example 2.3.3. Let $\mathcal{A}$ be an operator pre-C^*-algebra of the C^*-algebra A. Let $(E, D) \in \Psi^{(1)}_{mes}(\mathscr{A}, B)$ for c=some C^*-algebra B. Consider a smooth system $M_2(\mathscr{A}) = (M_2(\mathcal{A}) \hookrightarrow M_2(A))$. Then, for $a \in \mathcal{A}$, we have that

$$(E \oplus E, \left(\begin{smallmatrix} D & 0 \\ 0 & D+b \end{smallmatrix}\right)) \in \Psi^{(1)}_{mes}(M_2(\mathscr{A}), B)$$

for some odd operator b on E. Now, we consider a projector $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ on $M_2(\mathcal{A})$, and construct an unbounded $(A, M_2(A))$-KK-cycle given by $(pM_2(A), 0, \nabla)$, with ∇ being the Grassmanian connection. Observe that $pM_2(A) = A^2$ where A^2 is regarded as column vector, and we define the action of A on A^2 in a standard way, i.e $a(a_1, a_2)^\intercal = (aa_1, aa_2)^\intercal$. Now, we observe that $pM_2(A) \tilde{\otimes}_A (E \oplus E) \approxeq E$, and using this identification ti may be calculated directly, that

$$1_{pM_2(A)} \otimes_\nabla \left(\begin{smallmatrix} D & 0 \\ 0 & D+b \end{smallmatrix}\right) = D$$

or, in other words

$$(E \oplus E, \left(\begin{smallmatrix} D & 0 \\ 0 & D+b \end{smallmatrix}\right)) \mapsto (E, D)$$

From the other hand, there is a unitary equivalence

$$(E \oplus E, \left(\begin{smallmatrix} D & 0 \\ 0 & D+b \end{smallmatrix}\right)) \sim_u (E \oplus E, \left(\begin{smallmatrix} D+b & 0 \\ 0 & D \end{smallmatrix}\right))$$

given by the unitary $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on $E \oplus E$. However

$$(E \oplus E, \left(\begin{smallmatrix} D+b & 0 \\ 0 & D \end{smallmatrix}\right)) \mapsto (E, D + b)$$

But D and $D + b$ should not necessarily be unitary equivalent. So, we may fall into situation where unitary equivalent operators are mapped to not necessarily unitary equivalent by map defined by the same data (E, T, ∇).

We are still not in the position to discuss what can happen in the case when we switch to countably generated modules and genuine unbounded operators on them. This will most probably impose additional requirements also on the smooth systems. A more detailed study of this question will require further research.

2.3.2 *cb*-Isomorphism Classes of Operator Spaces

As we have seen in previous subsections, most of the results gave us the characterizations of operator spaces and operator algebras only up to a *cb*-isomorphism. From the other hand, a *cb*-isomorphisms preserved desirable structures, like almost riggedness of the modules. Therefore it seems that sometimes regarding the *cb*-isomorphism classes of operator space may spare some additional work.

Let X be an operator space. We denote by $[X]$ the class of operator spaces X_ω which are *cb*-isomorphic to X. Observe that if $f\colon X \to Y$ is a *cb*-map, then f induces a *cb*-map $f_{\omega_1,\omega_2}\colon X_{\omega_1} \to Y_{\omega_2}$ for any $X_{\omega_1} \in [X]$ and $Y_{\omega_2} \in [Y]$. Thus we may define a *cb-map* [f] between the *cb*-isomorphism classes of operator spaces. Denote

Moreover, for any two such spaces X_{ω_1} and Y_{ω_2} there is a *cb*-isomorphism between $\mathrm{CB}(X,Y)$ and $\mathrm{CB}(X_{\omega_1}, Y_{\omega_2})$. Indeed, the space $\mathrm{CB}(\mathcal{L}_1, \mathcal{L}_2)$ is an operator space [31]. Let ι_1, ι_2 be the complete isomprphisms $\iota_1\colon X_{\omega_1} \to X$ and $\iota_2\colon Y_{\omega_2} \to Y$. Then we can construct a map

$$\begin{aligned} \eta\colon \mathrm{CB}(X_{\omega_1}, Y_{\omega_2}) &\to \mathrm{CB}(X,Y) \\ f' &\mapsto \iota_2 f' \iota_1^{-1} \end{aligned}$$

By the construction, this map is *cb*. An inverse *cb*-isomophism is constructed analogously.

Hence we may speak about *cb*-maps between *cb*-isomorphism classes of operator spaces.

We have an analogous characterization of *cb*-isomorphism classes of operator pseudoalgebras, with *cb*-homomorphisms generating maps between the classes. By theorem 1.2.8 for any operator pseudoalgebra A there is also a genuine operator algebra in $[A]$.

Observe that if A has a pseudo-involution f, then

- By Theorem 1.2.16 there is an operator algebra in $A_\omega \in [A]$ such that the pseudo-involution on A induces a completely isometric involution $\iota_\omega f \iota_\omega^{-1}$.
- For any $A_\omega \in [A]$ the map $\iota_\omega f \iota_\omega^{-1}$ is a pseudo-involution.

Therefore, if A is an operator pseudoalgebra with a distinguished pseudo-involution f, then the *cb*-isomorphism class $[A]$ may be called *involutive*.

As for the operator modules, we have observed that a $\mathbb{C}$-linear *cb*-isomorphism $X \to Y$ induces a *cb*-isomorphism of $\mathrm{CB}(X) \to \mathrm{CB}(Y)$. Therefore, for an isomorphism class of operator algebra $[A]$ we may consider an *A-module cb-isomorphism class* $[X]$ as a set of all such A-modules X that there is a A-module *cb*-isomorphism between them, and the *$[A]$-module cb-isomorphism class* $[X]$ is defined analogously.

Now, we observe that most of the notions we have introduced in the paper are defined *up to a cb-isomorphism*. Indeed, for instance, if A and B are *cb*-isomorphic, then any almost rigged A-module E may be regarded as an almost rigged module over B and vice versa. Also, if E is a rigged module over A and E' is cb-isomorphic to E, then E' is obviously an almost rigged A-module. Therefore we lose nothing if we replace almost rigged modules over operator algebras by cb-isomorphism classes of rigged modules over cb-isomorphism classes of operator algebras.

An analogous observation holds for stuffed modules and cb-stabilizable modules. Moreover, using the cb-isomorphism classes of operator algebras and modules may sometimes make a picture even more uniform as in the case when we work with concrete objects. This approach may be useful by several considerations.

Recall that, as we have indicated in the discussion after the Theorem 1.2.54, the involution induced on $\mathrm{CB}^*_{\mathcal{A}^{(n)}}(\mathcal{E}^{(n)})$ by the one on $\mathrm{CB}^*_A(E)$ should not necessarily be completely isometric. The switch to cb-siomorphism classes, although does not solve this complication, may be used to hinder it.

Then, as we have indicated in the Remarks 2.1.5 and 2.1.16, the definition of smooth systems may be modified to work with pseudoalgebras instead of algebras. Equivalent smooth systems in this case will be regarded as a same object. In view of the discussion we have had about the Mesland fréchetization, this may also give rise to a notion of smooth noncommutative topology for C^*-algebras. However, the notion still remains vague, and we address a more precise formulation to the further research.

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