The unknowns can be written explicitly,

\[
\hat{x}_1 = Q_{\hat{x}_1} A_1^T C_1^{-1} l_1 + Q_{\hat{x}_1} A_2^T C_2^{-1} l_1 + Q_{\hat{x}_1} C_2^{-1} l_2, \tag{6.94}
\]

\[
\hat{x}_2 = Q_{\hat{x}_2} A_1^T C_1^{-1} l_1 + Q_{\hat{x}_2} A_2^T C_2^{-1} l_1 + Q_{\hat{x}_2} C_2^{-1} l_2. \tag{6.95}
\]

The last terms of these equations can be written with the respective sub-matrices from Eq. (6.89) to Eq. (6.93),

\[
Q_{\hat{x}_1} C_2^{-1} l_2 = -Q_{\hat{x}_1} A_1^T C_1^{-1} A_2 l_2, \tag{6.96}
\]

\[
Q_{\hat{x}_2} C_2^{-1} l_2 = l_2 - C_2 A_2^T C_1^{-1} A_2 l_2 + C_2 A_2^T C_1^{-1} A_1 Q_{\hat{x}_1} A_1^T C_1^{-1} A_2 l_2. \tag{6.97}
\]

If the pseudo observations \( l_2 \) are determined based on the integrals Eq. (5.23) with a specific gravitational potential and the a-priori variance-covariance matrix \( C_2 \rightarrow 0 \) then it follows,

\[
\hat{x}_1 = (A_1^T C_1^{-1} A_1)^{-1} A_1^T C_1^{-1} (l_1 - A_2 l_2), \tag{6.98}
\]

\[
\hat{x}_2 = l_2. \tag{6.99}
\]

In that case, the amplitudes are fixed according to the gravitational potential used and only the boundary values are corrected. The result corresponds to the result of a dynamical orbit determination. Besides this strict dynamical restrictions, there are basically three different possibilities to influence the kinematical orbit determination by dynamical information and to come up with a reduced kinematical orbit determination modification,

1. introduction of an approximate force function for the determination of the values \( (d_i \cdots d_j) \), combined with an approximate a-priori variance-covariance matrix \( C_2 = C(d) \),

2. fixing only some orbit parameters \( (d_i \cdots d_j) \) by setting a low variance-covariance matrix for \( C_2 = C(d) \rightarrow 0 \),

3. down- or up-weighting of the a-priori variance-covariance matrix \( C_2 \) in relation to \( C_1 \).

Because of these various possibilities to adopt the character of the orbit to a variety of applications covering the spectrum from a pure kinematical on the one end to a pure dynamical orbit determination on the other end makes this approach very flexible.

Fig. 6.3 shows a flowchart of the computation steps. Because of the fact that the force function depends on the knowledge of the orbit, the procedure requires an iteration. Therefore, a kinematical orbit determination has to be performed first, and subsequently a reduced-kinematical modification can be performed as shown in the flowchart.
Observed arc given by:

\[ r(\tau_k) = \bar{r}(\tau_k) + d(\tau_k) \]

\( \tau_k \) with \( k = 1, 2, 3, \ldots, K, \quad \tau_k \in [0,1] \)

**Dynamical restrictions**

\[ \ddot{d}_i = -\frac{2\mu}{v', \tau - \mu'} \int \sin(v, \tau') a^i(\tau, r, \dot{r}) \; d\tau' \]

\[ v' \in [v_i, \ldots, v_j], \forall i,j \leq n \]

\[ r = a r(\tau) + b r(\tau) + S d_n \]

\[ d = S' d = S' d + S' d' = \ddot{d} + d \]

\( \ddot{d} \) contains dynamical restrictions

\( d \) contains free coefficients

**Determination of the Euler-Bernoulli coefficients up to degree \( J_{\text{max}} \)**

from the discrete satellite's positions by a least squares adjustment

- Eq. (5.135)

\[ p = (PNN^TP)^{-1}PNd_n \]

\[ P = \begin{pmatrix} e_{3/J_{\text{max}}} & \cdots & e_{n/J_{\text{max}}} \\ b_{2/J_{\text{max}}} & \cdots & b_{2n/J_{\text{max}}} \end{pmatrix}, \quad N = \begin{pmatrix} I^1 & \cdots & I^n \\ \cdots & \cdots & \cdots \\ I_{2/J_{\text{max}}} & \cdots & I_{2n/J_{\text{max}}} \end{pmatrix}, \quad d_n = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \]

**Determination of the Fourier coefficients up to degree \( n \)**

from the discrete satellite's positions by a least squares adjustment

- Eqs. (6.83), (6.84), (6.94) and (6.95)

\[ \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad C_i = \begin{pmatrix} C_i \\ 0 \end{pmatrix} \]

\[ l_1 = \begin{pmatrix} r(\tau_k) \\ r(\tau_k) \end{pmatrix}, l_2 = \begin{pmatrix} \ddot{d} \\ \ddot{d} \end{pmatrix}, A = (a \ b), A = S^T \]

\[ x_1 = \begin{pmatrix} r(\tau_k) \\ \dot{r}(\tau_k) \end{pmatrix}, x_2 = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \]

**Determination of the coefficients of the residual Fourier series up to degree \( n' \)**

\[ \begin{pmatrix} \ddot{d}(\tau_k) \\ \ddot{d}(\tau_k) \end{pmatrix} = \ddot{d} = S'd_n - T^T \begin{pmatrix} E_{1/J_{\text{max}}} & \cdots & E_{n/J_{\text{max}}} \\ b_{1/J_{\text{max}}} & \cdots & b_{2n/J_{\text{max}}} \end{pmatrix} \begin{pmatrix} e_{3/J_{\text{max}}} \\ b_{2/J_{\text{max}}} \end{pmatrix}, \quad d_n = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \]

**Solution**

\[ r(\tau) = \bar{r}(\tau) + d'_{\mu}(\tau) + \ddot{d}(\tau) \]

\[ \bar{r}(\tau) = \frac{\sin \mu (1 - \tau)}{\sin \mu} r(\tau) + \frac{\sin \mu \tau}{\sin \mu} r(\tau) \]

\[ d'_{\mu}(\tau) = \sum_{j=1}^{n} e_{j} E_j(\tau) + \sum_{j=1}^{n} b_{j} B_{j+1}(\tau) \]

\[ \ddot{d}(\tau) = \sum_{j=1}^{n} \ddot{d}_j \sin(\nu \tau) \]

**Iteration, if necessary**

**Figure 6.3:** Computation scheme of the reduced-kinematical orbit determination strategy, with Euler-Bernoulli calculated from amplitudes of Fourier series.
7. Test Computations

7.1 GPS-SST Observation Preparations

7.1.1 SC7 Dataset

To determine the LEO’s orbits, code pseudo-range and carrier phase Satellite-to-Satellite Tracking (SST) measurements between a sufficient number of GNSS satellites and the LEO will be used in the following. To simulate the geometrical precise orbit determination procedure, the ephemerides of the GNSS satellites and the LEO are necessary. In this research, the simulated SC7 data set (Special Commission of IAG-SC7, www.geod.uni-bonn.de) has been used. In this data set, the mission scenarios of CHAMP, GRACE and GOCE as LEOs have been simulated to provide a unique easily accessible data set for various comparisons. The SC7 data set covers a time period of 30 days and includes the absolute positions, velocities and accelerations of the GPS satellites as well as the LEOs and reference frame specifications. The simulation scenarios are simplified in so far as there is no noise in the data for the LEOs and GPS orbits. The files of the GPS-satellites and CHAMP contain observation epochs in Modified Julian Date (MJD), absolute positions (r), velocities (˙r) and accelerations (¨r) of LEOs and GPS satellites. The absolute coordinates refer to a (quasi) inertial system with origin in the center of mass of the Earth and the axes directed to the principal axes of inertia. The relation of this quasi-inertial system to the Earth fixed coordinate system is defined by the sidereal time as rotation angle between the two specified coordinate systems.

7.1.2 GPS-SST Observation in the RINEX

Standard format for the GPS observations in case of ground-tracked and space-tracked GPS satellites is the Receiver INdependent Exchange (RINEX) format. To unify the simulation cases with the real cases and to enable a simple processing of the GPS data, the GPS observation file for the simulated observations have to be provided in the RINEX format. To mitigate the effect of low elevated GPS satellites on the accuracy of a point-wise positioning of LEOs, a proper cut-off angle is necessary in the processing. The elevation angle of the GPS receiver on-board LEO has to be applied in the data preparation of the simulated observations.

To generate the observation file which contains the GPS-SST observations in the simulated case, the ephemerides of the GPS satellites and the LEOs have been selected with an interval of 30 sec. Some comparisons are performed with a 10 sec. sampling rate. With the application of a $15^\circ$ cut-off angle, the code pseudo-range observations as GPS-SST observations read in the RINEX format as shown in Fig. 7.1. In this observation file, only code pseudo-range observations between the GPS satellites (as sender) and GPS receiver on-board LEO (as receiver) have been simulated.

7.1.3 Simulation Scenarios

To test the proposed orbit determination strategies, different scenarios have been simulated in the undifferenced (zero difference) observation mode. The simulation of a geometrical scenario has been designed based on error free zero difference GPS-SST observations. Then errors (white noise) in the GPS-SST observations simulate the real geometrical orbit determination environment. The geometrically determined orbits will be used as pseudo observations together with the error variance-covariance matrices for the sub-sequent kinematical and reduced-kinematical orbit determination modes. But also the direct GPS-SST observations can be used as original observations for the different orbit determination modifications, as shown in the following.

The geometrical precise orbit determination strategy has been tested in the simulation case with the SC7 data set. For the tests based on real observations, data sets of CHAMP have been used, which are provided by the ISDC-GFZ Potsdam, Germany.
To test the geometrical orbit determination strategy to estimate the LEOs absolute positions from code pseudo-range and carrier phase observations in the zero differenced mode, a simulation example will be considered. Based on the SC7 data set, one satellite revolution has been selected and carrier phase observations have been simulated. In the pre-processing step, a cut off angle of 15° has been applied to the GPS-SST observations to simulate the real case of a geometrical orbit determination. The GPS-SST observations are contaminated with white noise of a standard deviation of 1 cm, 2 cm and 5 cm, respectively. Figure 7.1 shows a simulated CHAMP orbit. Figure 7.2 shows the number of processed GPS satellites and the position dilution of precision (PDOP) is shown. This gives an impression of the geometrical strength of the GPS satellite configuration. There are always at least 5 GPS satellites available and occasionally up to 10 GPS satellites. The PDOP remains between 1 and 1.5, which represents a strong geometrical satellite configuration. In Fig. 7.4, elevation of the GPS satellites and PDOP values at every observation epoch are shown. It should be pointed out that PDOP values are used to estimate the position precision.

The results of the processing of the GPS-SST observations in the simulation case are shown in Fig. 7.6 as differences of the point-wise estimated and given absolute positions. In Fig. 7.5, the corresponding GPS-SST observation residuals are presented. As can be seen, the differences between the estimated LEO absolute positions and the given absolute positions are small.
7.2. Geometrical Precise Orbit Determination (GPOD)

To test the proposed geometrical LEO orbit determination strategy in the error case, the GPS-SST observations have to be contaminated with different white noise levels. In the following white noise levels of 1cm, 2cm and finally 5cm in the GPS-SST observations have been selected. In Fig. 7.7, the absolute position differences referred to the SC7 positions are shown for a white noise of 1cm. As can be seen, most of the differences are in the range of the observation noise of 1cm. It should be pointed out that in the GPS data processing, especially in zero difference mode of data processing, there is a linear dependency between the observation precision and the estimated absolute position precision. For example, if the GPS observations are preprocessed and the precision of observations is in the range of 1cm, then the precision of estimated absolute position is in the range of 1cm as well. But it is clear that the configuration of tracked GPS satellites plays an important role in the precision of the estimated positions as well (which is expressed by the PDOP).
value). Therefore, the PDOP value is a factor, which indicates the factor of precision of the position (Xu 2007). It can reduce the accuracy of the estimated positions in case of a high PDOP value. In Fig. 7.8 the GPS-SST observation residuals are shown for the white noise case of 1cm. Figs. 7.9 and 7.10 show the situation for a white noise level of 2cm. Nowadays, the accuracy of the estimated LEO orbit is in the size between 2cm to 5cm. Figs. 7.11 and 7.12 show the corresponding results for a white noise level of 5cm.

It should be mentioned that the geometrical orbit determination produces an ephemeris of position coordinates where the accuracy depends on the configuration of the GPS satellites and on the precision of the GPS-SST observations. For the simulation cases, the processing of the GPS-SST observations can be performed either in the single epoch processing mode or in the batch processing mode. The single and batch processing modes have to deliver the same results.

Figure 7.7: The GPS-SST observation residuals (white noise of 1cm).

Figure 7.8: Point-wise estimated CHAMP orbit minus SC7 absolute positions (white noise of 1cm).

Figure 7.9: The GPS-SST observation residuals (white noise of 2cm).

Figure 7.10: Point-wise estimated CHAMP orbit minus SC7 absolute positions (white noise of 2cm).

Figure 7.11: The GPS-SST observation residuals (white noise of 5cm).

Figure 7.12: Point-wise estimated CHAMP orbit minus SC7 absolute positions (white noise of 5cm).
7.2.2 Analysis of Real GPS-SST Observations

The precise absolute positions of the LEOs can be determined either from the carrier phase GPS-SST observations or from the subsequent time differenced carrier phase GPS-SST observations at two sequential epochs in the batch processing mode. If the difference between two carrier phase GPS-SST observations at subsequent epochs has been built and the observation epochs are close together, then the GPS ambiguity parameters are canceled out. Additionally, many errors in the ZD mode of GPS-SST observations can be eliminated or mitigated with the corresponding models (e.g. ionosphere error, antenna phase center offsets and its variations, multi-path, etc.). Therefore, besides the batch processing of carrier phase GPS-SST observations, the sequential time differenced carrier phase observations have been proved to be very efficient for the geometrical LEO precise orbit determination. To determine the geometrical LEO precise orbits, in the real case, initial LEO absolute positions with an accuracy of a few meters have to be estimated based on the Bancroft method with the code pseudo-ranges as a first step. These absolute positions can be used subsequently as initial values for the LEOs absolute positions based on the ionosphere-free code pseudo-range observations to reduce the number of iterations in the data processing. To avoid the GPS satellites with low elevations, at first, a 15° elevation cut-off angle has to be applied to the GPS-SST observations. With the estimated absolute positions in the code pseudo-range process and with the help of the GPS receiver clock offset between two subsequent epochs, the outliers (or cycle slips) are detected and removed in the iteration process. In this method, the estimated LEO orbit is determined point-wise. The geometrical configuration (expressed by the Position Dilution Of Precision, PDOP) of the GPS satellites plays an important role for the GPS data processing accuracy.

To verify the proposed geometrical orbit determination procedure, a 30 minute arc of CHAMP (2002 07 20 12h 48m 00.0s - 2002 07 20 13h 18m 00.0s) with an interval rate of 30 sec. has been selected. The CHAMP ground track is shown in Fig. 7.13(b). The number of the GPS satellites, the corresponding PDOPs and the elevations of the GPS satellites are shown in Figs. 7.14 and 7.15, respectively. In a first step, the approximate estimation of absolute LEO positions are determined based on the Bancroft method (BANCROFT 1985). Fig. 7.16 shows the absolute position differences and Fig. 7.17 the corresponding code pseudo-range observation residuals. The precision of the estimated LEO positions are in the range of a few meters.

![Figure 7.13: The ground track of the 30 minute CHAMP short arc for the time: (a) 2002 03 21 13h 30m 0.0s–14h 00m 0.0s (b) 2002 07 20 12h 48m 0.0s–13h 18m 0.0s (c) 2003 03 21 17h 20m 0.0s–17h 50m 0.0s (d) 2003 03 31 17h 00m 0.0s–17h 30m 0.0s.](image)

To improve the accuracy of the LEO positions, the approximate positions are used as initial values to screen the GPS-SST observations for outliers (as described in chapter 3). Then they are used as initial values to estimate the LEO absolute positions with the code pseudo-range GPS-SST observations. In Figs. 7.18 and 7.19, the absolute position differences and the observation residuals are shown. The position differences are in the range of the accuracy of the code pseudo-range GPS-SST observations in the size of smaller than two meter.
Final improvements can be performed based on the carrier phase GPS-SST observations between the GPS satellites and the GPS receiver on-board LEO. The carrier phase observations are very precise (in the size of \( \text{mm} \)) and play an important role in the precise point positioning (PPP or 3P), either in the static or in the kinematical mode of the geometrical point positioning.

Because of the fact that the accuracy of carrier phase observations are in the range of some millimeters, one can expect millimeter accuracy of GPS receiver positions. In case of GPS receivers at the ground stations, the problematic modeling of atmospheric parameters can reduce the accuracy of the estimated positions. The situation is less critical for GPS receivers at satellite altitude. There is no troposphere in the altitude of LEOs, but at this altitude the ionosphere can affect the GPS-SST observations. The multi-path is another problem in the data processing. Therefore, the LEO absolute positions based on the carrier phase GPS-SST observations can be estimated with an accuracy of 2 to 5cm. The absolute position differences with respect to the CHAMP PSO dynamical orbit provided by GFZ as well as the corresponding GPS-SST residuals are shown in Fig. 7.20(b) to Fig. 7.23(b).
To validate the geometrically determined point-wise CHAMP orbits, different short arcs of 30 minutes are selected. In Figs. 7.20(b) and 7.20(c), the CHAMP absolute position differences with respect to the PSO CHAMP orbit and the dynamical orbits (provided by Švehla, 2003) are shown for the time window of 2002 03 21 13h 30m 0.0s–14h 00m 0.0s. Fig. 7.20(d) shows the position differences of the PSO CHAMP dynamical orbit with respect to Švehla’s dynamical orbit and Fig. 7.20(a) the carrier phase GPS-SST observation residuals. The difference plots show a clear systematic deviation pattern caused by the fact that the GFZ PSO as well as Švehla’s solution are based on dynamical models. If these systematic effects are removed by a polynomial of degree 5, then the trend reassessed RMS values can be determined. The trend reassessed RMS values for the three coordinates \( x, y \) and \( z \) are given for the comparisons \( b, c \) and \( d \). To show the validation of geometrically determined CHAMP orbit, another three examples at different times are selected. Figs. 7.21(a), 7.21(b), 7.21(c) and 7.21(d) show the carrier phase GPS-SST observation residuals and the position differences for the different CHAMP reference orbits for the time window 2002 07 20 12h 48m 0.0s–13h 18m 0.0s. An approximate trend reassessed RMS value of about 1cm is achieved for the CHAMP absolute positions with respect to the PSO CHAMP orbit. Figs. 7.22(a) to 7.23(d) show the carrier phase GPS-SST observation results and position differences for the time windows 2003 03 21 12h 00m 0.0s–12h 30m 0.0s and 2003 03 31 17h 00m 0.0s–17h 30m 0.0s, respectively. In all cases, the trend reassessed RMS values with respect to PSO and Švehla’s dynamical orbits are in the range of 1 to 2cm, but the GPS-SST observation residuals show the smallest observation residuals.
**Figure 7.20:** Geometrically determined CHAMP arc (a) carrier phase GPS-SST observation residuals (b) position differences: IGG solution - CHAMP PSO orbit (c) IGG solution - TUM dynamic solution (Švehla, 2002) (d) TUM dynamic solution (Švehla, 2003) - CHAMP PSO orbit for the time 2002 03 21 13h 30m 0.0s–14h 00m 0.0s.

**Figure 7.21:** Geometrically determined CHAMP arc (a) carrier phase GPS-SST observation residuals (b) position differences: IGG solution - CHAMP PSO orbit (c) IGG solution - TUM dynamic solution (Švehla, 2002) (d) TUM dynamic solution (Švehla, 2003) - CHAMP PSO orbit for the time 2002 07 20 12h 48m 0.0s–13h 18m 0.0s.
7.2. Geometrical Precise Orbit Determination (GPOD)

Figure 7.22: Geometrically determined CHAMP arc (a) carrier phase GPS-SST observation residuals (b) position differences: IGG solution - CHAMP PSO orbit (c) IGG solution - TUM dynamic solution (Švehla, 2002) (d) TUM dynamic solution (Švehla, 2003) - CHAMP PSO orbit for the time 2003 03 21 17h 20m 0.0s–17h 50m 0.0s.

Figure 7.23: Geometrically determined CHAMP arc (a) carrier phase GPS-SST observation residuals (b) position differences: IGG solution - CHAMP PSO orbit (c) IGG solution - TUM dynamic solution (Švehla, 2002) (d) TUM dynamic solution (Švehla, 2003) - CHAMP PSO orbit for the time 2003 03 31 17h 00m 0.0s–17h 30m 0.0s.
7.3 Kinematical Precise Orbit Determination (KPOD)

As described in Sec. 6.1, the proposed kinematical orbit determination approach can be interpreted as continuous representation of the LEO’s arc. The continuous representation of the satellite’s arc is realized by a combination of the Euler-Bernoulli polynomials up to degree $J_{\text{max}}$ and the sine series up to index $\bar{n}$. This procedure can be performed either based on the GPS-SST observations directly or based on the LEO’s geometrically determined absolute positions. Both methods have to deliver the same results if the full variance-covariance information (with the correlation information between the points in case of positions as pseudo observations) are used in both cases. To verify the proposed kinematical precise orbit determination strategy, different scenarios are simulated based on the SC7 data set. After the verification, the proposed strategy is used for the real case, i.e. the CHAMP’s data which are provided by the ISDC-GFZ, Potsdam.

7.3.1 Simulated Case

The orbit determination approach is tested based on a simulation scenario for a 30 minute arc of a simulated CHAMP orbit, adapted from the SC7 simulation scenarios as described in Sec. 7.1. To be as close as possible to the reality, a 15° cut-off angle has been chosen for the measurements between the GPS receiver, fixed to CHAMP, and the GPS satellites. In a first test based on simulated data, the error-free case is investigated (Sec. 7.3.1.1) and, in a second simulation test, the case of a white noise of 2cm for the GPS-SST measurements to derive positions so that they are used as pseudo-observations for the subsequent kinematical orbit determination procedure (Sec. 7.3.1.2). In the case of high-low GPS-SST observations (e.g. as carrier phase observations) for the kinematical orbit determination, a white noise corresponding to 2cm pseudo-range accuracy has been modeled. This GPS-SST observation error produces also an error of approximately 2cm in each of the position coordinates. The Table 7.1 shows the various simulation examples. Despite the fact that the computation procedure shown in Fig. 6.1 and Fig. 6.2 are very similar, they are considered separately. For each computation alternative, the observations for the determination of the kinematical orbit are considered either as positions (introduced as pseudo observations) or directly as carrier phase GPS-SST observations. In many applications of kinematical orbits, the position coordinates derived from the geometrical orbit determination procedure are used instead of the high-low GPS-SST observations. Therefore, the difference of both solutions shall be discussed.

| Table 7.1: Summary of all kinematical precise orbit determination cases. |
|---|---|---|---|---|---|
| error case | noise free | white noise 2cm |
| computation type | Fig. 6.1 | Fig. 6.2 | Fig. 6.1 | Fig. 6.2 |
| observation type | SST | Pos. | SST | Pos. | SST | Pos. |
| Figure | Fig.7.24 | Fig.7.25 | Fig.7.26 | Fig.7.27 | Fig.7.28 | Fig.7.29 | Fig.7.30 | Fig.7.31 |

7.3.1.1 Noise Free Case

In a first step, the GPS-SST observations with the corresponding variance-covariance information are used to estimate the kinematical precise orbit of CHAMP as described in the flowchart of Fig. 6.1. As shown in Fig. 6.1, the LEO arc boundary positions and the sine amplitudes are estimated based on the GPS-SST observations. The Euler-Bernoulli polynomial coefficients up to degree $J_{\text{max}} = 4$ are estimated independently from the GPS-SST observations as well. In a first step, the residual sine series are estimated by a least squares estimation. Figs. 7.24(a) to 7.27(a) show the absolute position differences at a time interval of 10sec., while the orbit determination is based on a sampling rate of 30sec. The upper limit of the sine series are chosen as $\bar{n} = 20, 30, 30, \text{and } 59$. The latter case represents the interpolation case, that means no redundancy exists.

In judging the results in the following figures, we want to point out that the orbits are determined from observations sampled at a 30 seconds rate while the comparisons with the true values are performed at every
10 seconds. This is the reason that we see RMS values different from zero also in the interpolation cases. To avoid the influence of remaining Gibb’s effects in the RMS results as shown in the RMS tables to each orbit determination example, the boundary sections in the size of 5% of the total arc are blanked out. We will see later that this is important only in the error-free cases; in case of noisy observations the Gibb’s effects are not visible anymore compared to the general errors in the orbit determination results (compare the Fig. 7.32 for the error-free case and the Fig. 7.33 for the noise case).

The RMS table of Fig. 7.24 shows, as expected, that the best result is achieved for the interpolation case. As mentioned before, the RMS values represent the differences of the computed coordinates and the true values given at every 10 second. This holds also for the velocities (shown in Fig. 7.24(b)) and for the accelerations (shown in Fig. 7.24(c)). The velocities and accelerations can be determined by analytical differentiation as outlined in Sec. 5.3.3. Especially, the interpolation cases demonstrate the high accuracy of the orbit representation by the mixed series approximation of sine series and Euler-Bernoulli polynomials. As expected, these differences will be larger in the error examples as discussed in Sec. 7.3.1.2.

If the point-wise LEOs positions are determined with the geometrical orbit determination strategy based on the GPS-SST observations (see Chapter 4), then the estimated absolute positions and their variance-covariance can be used as pseudo-observations in the kinematical data processing procedure. As shown in Fig. 6.1, the boundary positions as well as the sine amplitudes are estimated based on the point-wise absolute positions with their variance-covariance information. The Euler-Bernoulli coefficients up to degree $J_{\text{max}} = 4$ are estimated from the geometrically determined positions independently. Finally, the residual sine amplitudes are estimated based on the sine analysis of the residual function. The LEO orbit can be represented with the boundary positions, the Euler-Bernoulli polynomials up to degree $J_{\text{max}} = 4$ and the residual sine series (with the upper indices 20, 30, 40 or 59).

The results shown in Fig. 7.25 correspond to the results shown in Fig. 7.24 with the only difference that they are derived from a position ephemeris with a 30sec. sampling rate, derived geometrically in a prior step from the GPS-SST observations. The variance-covariance matrices of the three coordinates x, y and z of the absolute positions are used in the kinematical orbit determination procedure as well. The results are similar to those of Fig. 7.24, with the minimal RMS values in the positions, velocities and accelerations for the interpolation case. The comparison of the cases shown in the Figs. 7.24 and 7.25 show that there are slightly smaller RMS values for the case where the positions and its variance-covariance matrices are introduced in the computation procedure as pseudo observations than in case of using directly the GPS-SST observations.

The flowchart in Fig. 6.2 shows the procedure, where the Euler-Bernoulli polynomial coefficients are estimated based on the sine series amplitudes. In the other words, the Euler-Bernoulli coefficients are estimated in the spectral domain. This procedure can again be performed either based on the GPS-SST observations or based on the geometrically determined LEO absolute positions. The results shown in the Figs. 7.26 and 7.27 correspond to the results shown in the Figs. 7.24 and 7.25 with the only difference that the Euler-Bernoulli polynomials are determined from the amplitudes of the sine series as shown in the flowchart of Fig. 6.2. This procedure differs slightly from the procedure as shown in the flowchart of Fig. 6.1, because in the case of Fig. 6.2, the Euler-Bernoulli coefficients are determined implicitly also under a simultaneous correction of the boundary vectors of the short arc. The comparison of the results in the Figs 7.26 and 7.27 shows identical results (at least in the RMS values of the positions, velocities and accelerations) with the minimal RMS values for the interpolation case (as expected) and also slightly better than the results shown in Fig. 7.25.

Summarizing the results of the error-free simulation cases, we conclude that the best case with minimal RMS values for a 10sec. ephemeris (in the positions, velocities and accelerations) based on an observation series with a 30 sec sampling rate is that of using either GPS-SST observations directly or positions as pseudo observations indirectly by following the determination procedure as shown in the flowchart of Fig. 6.2. Obviously, it is preferable to derive the Euler-Bernoulli series from the sine series coefficients, which are determined in a prior least squares adjustment step by simultaneous correcting the boundary values together with the sine coefficients.
Figure 7.24: (a) Position differences (b) velocity differences (c) acceleration differences (d) Fourier spectrum (reference motion: ellipse mode, $J_{max}=4$, $N_F=300$, error free case, observation type: SST, method: flowchart Fig. 6.1, observation sampling rate: 30 sec., test sampling rate: 10 sec., without $\approx$ 5% at boundaries $\equiv$ 9 points or 80 sec.).

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4. Kinematical Precise Orbit Determination (KPOD)

Figure 7.25: (a) Position differences (b) velocity differences (c) acceleration differences (d) Fourier spectrum (reference motion: ellipse mode, \( J_{\text{max}} = 4 \), \( N_F = 300 \), error free case, observation type: Positions, method: flowchart Fig. 6.1, observation sampling rate: 30 sec., test sampling rate: 10 sec., without \( \approx 5\% \) at boundaries \( \equiv 9 \) points or 80 sec.).

\[
\begin{array}{c|c|c|c}
\text{index} & \text{Pos.} (m) & \text{Vel.} (m/s) & \text{Acc.} (m/s^2) \\
20 & 0.008496 & 0.000319 & 0.000012 \\
30 & 0.002708 & 0.000144 & 0.000008 \\
40 & 0.001177 & 0.000081 & 0.000006 \\
59 & 0.000126 & 0.000014 & 0.000001 \\
\end{array}
\]
Figure 7.26: (a) Position differences (b) velocity differences (c) acceleration differences (d) Fourier spectrum (reference motion: ellipse mode, $J_{\text{max}}=4$, $N_F=300$, error free case, observation type: SST, method: flowchart Fig. 6.2, observation sampling rate: 30 sec., test sampling rate: 10 sec., without $\approx 5\%$ at boundaries $\equiv 9$ points or 80 sec.).

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