Then the derivatives $d^{(2j)}(t)$ at $t_A$ and $t_B$ up to the 12th order are determined from the Eqs. (5.84) and (5.85) and compared to the analytically or numerically precisely determined derivatives $d^{(2j)}(t_A)$ and $d^{(2j)}(t_B)$.

A simple first example of an orbit is a circular orbit. In this case, the derivatives can be derived analytically and, therefore, the numerical values can be considered as error free "true" values. Here, a circular orbit with radius $r=6774383.56175931$ m and an arc length of 30 minute has been selected. The arc is located in the $y-z-$ plane. To derive the arc derivatives from the coefficients of the Euler and Bernoulli polynomials, the 30 minute arc is approximated by an Euler-Bernoulli polynomial with a maximum index $J_{\text{max}}=6$, $d_6^P(\tau)$.

As reference motion, the "straight line" modification has been selected. The approximation quality of the circular arc by the Euler-Bernoulli polynomials is shown in Fig. 5.22(a). The residuals show an oscillating structure as expected but the mean deviations are very small and, therefore, the polynomial coefficients (Table 5.1) can be considered as reliable. The derivatives $d^{(2j)}(t_A)$ and $d^{(2j)}(t_B)$ are given in Table 5.2. This table shows that the derivatives become not acceptable from the index $j=6$ upwards. Because of the fact that the "true" values are error-free, in this case the deviations are caused by the restricted numerical accuracy of the determination of the Euler-Bernoulli polynomial coefficients derived from the least squares fit.

Another example is a Keplerian orbit. In this case, the derivatives can be derived only numerically but with a very high reliability as mentioned above and, therefore, the numerical values can be considered also as error free "true" values, compared to the derivatives derived from the polynomial coefficients. Again, a 30 minute arc of the (nearly circular) Keplerian orbit has been selected. To determine the derivatives from the coefficients of the Euler and Bernoulli polynomials, the 30 minute arc is again approximated by an Euler-Bernoulli polynomial with a maximum index $J_{\text{max}}=6$, $d_6^P(\tau)$. The approximation quality of the Keplerian
Table 5.1: Coefficients of the Euler and Bernoulli polynomials for the circular orbit (reference motion: straight line).

<table>
<thead>
<tr>
<th>j</th>
<th>( e_{2j} )</th>
<th>( b_{2j+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000000E+00</td>
<td>-0.386570E+07</td>
</tr>
<tr>
<td>2</td>
<td>0.000000E+00</td>
<td>0.138320E+07</td>
</tr>
<tr>
<td>3</td>
<td>0.000000E+00</td>
<td>-0.185296E+06</td>
</tr>
<tr>
<td>4</td>
<td>0.000000E+00</td>
<td>0.137653E+05</td>
</tr>
<tr>
<td>5</td>
<td>0.000000E+00</td>
<td>-0.593884E+03</td>
</tr>
<tr>
<td>6</td>
<td>0.000000E+00</td>
<td>0.373499E+02</td>
</tr>
</tbody>
</table>

Table 5.2: Derivatives of a circular arc at the arc boundaries: derived from the Euler-Bernoulli polynomial coefficients and the analytically derived values (reference motion: straight line).

<table>
<thead>
<tr>
<th>j</th>
<th>( r^{(2j)}(t_A) ) derived</th>
<th>( r^{(2j)}(t_B) ) derived</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000000E+00</td>
<td>0.000000E+00</td>
</tr>
<tr>
<td>2</td>
<td>0.000000E+00</td>
<td>0.111359E-04</td>
</tr>
<tr>
<td>3</td>
<td>0.000000E+00</td>
<td>-0.142775E-10</td>
</tr>
<tr>
<td>4</td>
<td>0.000000E+00</td>
<td>0.183142E-16</td>
</tr>
<tr>
<td>5</td>
<td>0.000000E+00</td>
<td>-0.229393E-22</td>
</tr>
<tr>
<td>6</td>
<td>0.000000E+00</td>
<td>0.411080E-28</td>
</tr>
</tbody>
</table>

Table 5.3: Coefficients of the Euler and Bernoulli polynomials for the Keplerian orbit (reference motion: ellipse mode).

<table>
<thead>
<tr>
<th>j</th>
<th>( e_{2j} )</th>
<th>( b_{2j+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.12329E+05</td>
<td>0.128078E+03</td>
</tr>
<tr>
<td>2</td>
<td>-0.466065E+04</td>
<td>-0.704263E+02</td>
</tr>
<tr>
<td>3</td>
<td>0.558174E+04</td>
<td>0.238045E+02</td>
</tr>
<tr>
<td>4</td>
<td>-0.167750E+04</td>
<td>-0.530766E+01</td>
</tr>
<tr>
<td>5</td>
<td>0.693424E+03</td>
<td>0.153363E+01</td>
</tr>
<tr>
<td>6</td>
<td>0.900661E+02</td>
<td>0.221174E+00</td>
</tr>
</tbody>
</table>
5.4. Numerical Verifications

Figure 5.23: Spectrum of the Euler Bernoulli orbit based on the discrete Fourier analysis (reference motion: straight line mode, \(J_{\text{max}}=6\)).

Figure 5.24: Residuals of the Euler Bernoulli orbit analyzed by the discrete Fourier (reference motion: straight line mode, \(J_{\text{max}}=6\)).

Table 5.4: Derivatives of the Keplerian orbit at the arc boundaries: derived from the Euler-Bernoulli polynomial coefficients and the “true” values (reference motion: ellipse mode).

<table>
<thead>
<tr>
<th>(j)</th>
<th>(r^{(2j)}(t_A)) derived</th>
<th>(r^{(2j)}(t_B)) derived</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
<td>(z)</td>
</tr>
<tr>
<td>1</td>
<td>(0.643729E+01)</td>
<td>(-0.797184E-01)</td>
</tr>
<tr>
<td>2</td>
<td>(-0.825714E-05)</td>
<td>(0.101869E-06)</td>
</tr>
<tr>
<td>3</td>
<td>(0.105349E-10)</td>
<td>(-0.127994E-12)</td>
</tr>
<tr>
<td>4</td>
<td>(-0.130665E-16)</td>
<td>(0.149604E-18)</td>
</tr>
<tr>
<td>5</td>
<td>(0.191618E-22)</td>
<td>(-0.124952E-24)</td>
</tr>
<tr>
<td>6</td>
<td>(-0.729120E-28)</td>
<td>(-0.538715E-30)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(j)</th>
<th>(r^{(2j)}(t_A)) true</th>
<th>(r^{(2j)}(t_B)) true</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
<td>(z)</td>
</tr>
<tr>
<td>1</td>
<td>(0.643734E+01)</td>
<td>(-0.797181E-01)</td>
</tr>
<tr>
<td>2</td>
<td>(-0.825720E-05)</td>
<td>(0.101868E-06)</td>
</tr>
<tr>
<td>3</td>
<td>(0.105345E-10)</td>
<td>(-0.124952E-12)</td>
</tr>
<tr>
<td>4</td>
<td>(-0.131410E-16)</td>
<td>(0.149651E-18)</td>
</tr>
<tr>
<td>5</td>
<td>(0.147863E-22)</td>
<td>(-0.117397E-24)</td>
</tr>
<tr>
<td>6</td>
<td>(-0.747261E-29)</td>
<td>(-0.217763E-30)</td>
</tr>
</tbody>
</table>
Table 5.5: Coefficients of the Euler and Bernoulli polynomials for the orbit for a gravity field complete up to degree and order 20 (reference motion: ellipse mode).

<table>
<thead>
<tr>
<th>$j$</th>
<th>$e_{2j}$</th>
<th>$b_{2j+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td></td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>1</td>
<td>-0.207985E+05</td>
<td>-0.222995E+03</td>
</tr>
<tr>
<td>2</td>
<td>0.155647E+06</td>
<td>-0.837390E+04</td>
</tr>
<tr>
<td>3</td>
<td>0.235208E+07</td>
<td>-0.823831E+05</td>
</tr>
<tr>
<td>4</td>
<td>0.955080E+07</td>
<td>-0.370319E+06</td>
</tr>
<tr>
<td>5</td>
<td>0.669390E+07</td>
<td>-0.307651E+06</td>
</tr>
<tr>
<td>6</td>
<td>0.425513E+06</td>
<td>-0.200816E+05</td>
</tr>
</tbody>
</table>
5.4. Numerical Verifications

Figure 5.27: Spectrum of the Euler Bernoulli orbit based on the discrete Fourier analysis (reference motion: dynamical reference orbit $N_R=30$ mode, $J_{\text{max}}=6$).

Figure 5.28: Residuals of the Euler Bernoulli orbit analyzed by the discrete Fourier (reference motion: dynamical reference orbit $N_R=30$ mode, $J_{\text{max}}=6$).

Table 5.6: Derivatives of the orbit for a gravitational field complete up to degree and order 20 at the arc boundaries: derived from the Euler-Bernoulli polynomial coefficients and the "true" values (ellipse mode).

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\mathbf{r}^{(2j)}(t_A)$ derived</th>
<th>$\mathbf{r}^{(2j)}(t_B)$ derived</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>1</td>
<td>0.642626E+01</td>
<td>-0.796266E-01</td>
</tr>
<tr>
<td>2</td>
<td>-0.783392E-05</td>
<td>0.138104E-06</td>
</tr>
<tr>
<td>3</td>
<td>0.102924E-09</td>
<td>0.169859E-10</td>
</tr>
<tr>
<td>4</td>
<td>0.918537E-14</td>
<td>0.206370E-14</td>
</tr>
<tr>
<td>5</td>
<td>0.292770E-18</td>
<td>0.798495E-19</td>
</tr>
<tr>
<td>6</td>
<td>0.213899E-23</td>
<td>0.709214E-24</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\mathbf{r}^{(2j)}(t_A)$ true</th>
<th>$\mathbf{r}^{(2j)}(t_B)$ true</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.642595E+01</td>
<td>-0.796355E-01</td>
</tr>
<tr>
<td>2</td>
<td>-0.815142E-05</td>
<td>0.138104E-06</td>
</tr>
<tr>
<td>3</td>
<td>0.100836E-10</td>
<td>-0.313140E-12</td>
</tr>
<tr>
<td>4</td>
<td>0.626784E-16</td>
<td>0.583655E-16</td>
</tr>
<tr>
<td>5</td>
<td>-0.163904E-18</td>
<td>-0.142697E-19</td>
</tr>
<tr>
<td>6</td>
<td>0.139661E-21</td>
<td>0.272801E-23</td>
</tr>
</tbody>
</table>
are by the Euler-Bernoulli polynomials is shown in Fig. 5.22(b) (ellipse mode). The mean deviations are very small and, therefore, the polynomial coefficients can be considered as reliable. The deviations show an oscillating character as expected, and are more or less identical for the straight line mode and the ellipse mode of the reference motion. The Euler-Bernoulli coefficients and the derivatives \( r^{(2j)}(t_A), r^{(2j)}(t_B) \) are shown in Tables 5.3 and 5.4. These tables show that the derivatives become not acceptable already from the index \( j = 5 \) upwards. The reason might be the same as above.

If the degree of the gravitational field is increased, resulting in a "rough" orbit then the least squares fit of the series in terms of Euler-Bernoulli polynomials becomes increasingly worse. In these cases, the derivatives of the arc at the boundary epochs do not coincide well with the derivatives based on a very precise numerical differentiation. This can be observed already in case of the "Main problem". Here the Euler-Bernoulli polynomial fit is still sufficient (Fig. 5.36), but the derivatives differ already for \( J_{\text{max}} = 4 \) in contrast to the Keplerian orbit. In case of an orbit example determined for a gravitational field complete up to a spherical harmonic degree 20, the situation does not change very much. The derivatives differ again for an index \( J_{\text{max}} = 4 \) and upwards. The results are given in Tables 5.5 and 5.6 for the ellipse mode of the reference motion. Fig. 5.37 demonstrates that the approximation quality of the series in terms of the Euler-Bernoulli polynomials up to degree \( J_{\text{max}} = 6 \) is not better than a couple of centimeters.

5.4.2.2 Comparisons of Fourier Series Computations

The coefficients of the Fourier series convey directly a relation to the force function model, see e.g. Eq. (5.13). A similar relation is important also for the representation of the orbit in terms of Euler and Bernoulli polynomials. In Sec. 5.3.2.2, it is shown that the Fourier coefficients can be expressed in terms of the coefficients of the Euler and Bernoulli polynomials, e.g. theoretically by Eqs. (5.99) and (5.100), and as an example by Eq. (5.129) for an upper polynomial degree 5 (i.e. \( J_{\text{max}} = 2 \)), and finally for the general case in Eq. (5.132). It should be pointed out already here, that the series in terms of Euler-Bernoulli polynomials can be represented by an infinite series of sine coefficients. The approximation quality of this sine series to represent the real orbit is certainly limited by the approximation quality of the Euler-Bernoulli polynomials. In the following, the numerical properties of the direct Fourier analysis and the procedure based on a prior determination of the Euler-Bernoulli polynomial coefficients will be investigated.

**Sine analysis of the difference function:** According to Eq. (5.39), the solution series Eq. (5.9) or Eq. (5.19) represents a Fourier series of the difference function,

\[
d(\tau) = r(\tau) - \bar{r}(\tau) = \sum_{\nu=1}^{\infty} d_\nu \sin(\nu \pi \tau) = d_\nu^F(\tau).
\]

The Fourier coefficients \( d_\nu \) of the function \( d(\tau) \),

\[
d_\nu = 2 \int_{\tau' = 0}^{1} d(\tau) \sin(\nu \pi \tau') \, d\tau',
\]

can be directly determined by a Fourier analysis according to Eq. (5.51)

\[
d_\nu \approx \frac{2}{K + 1} \sum_{k=1}^{K} d(\tau_k) \sin \left( \frac{\nu \pi k}{K + 1} \right),
\]

if the function \( d(\tau) \) is given discrete at a regular sampling rate of \( \tau_k \) (see Eq. (5.52)),

\[
d(\tau_k) \quad \text{with} \quad \tau_k = \frac{k}{K + 1}, \quad k = 1, 2, 3, ..., K, \quad \tau_k \in ]0, 1[.
\]

Because of the orthogonality relation of the discretized sine functions as basis functions, the coefficients \( d_\nu \) up to an upper index of \( n = K \) can be determined rigorously in case of a regular equidistant sampling. In case of an upper summation index \( n \) and a remainder term \( R_F(\tau) \), we can write according to Eq. (5.94),

\[
d(\tau) = r(\tau) - \bar{r}(\tau) = d_\nu^F(\tau) + R_F(\tau) = d_\nu^\infty F(\tau),
\]
with the remainder term \( R_F(\tau) \) according to Eq. (5.95),
\[
R_F(\tau) = \sum_{\nu=n+1}^{\infty} d_{\nu} \sin(\nu \pi \tau).
\]
The minimal quadratic error for the coordinates reads in this case according to Eq. (5.48),
\[
\int_0^1 (d_i(\tau) - F_i(\tau))^2 d\tau = \int_0^1 R_{F_i}^2 d\tau = \int_0^1 (d_i(\tau))^2 d\tau - \frac{1}{2} \sum_{\nu=1}^{n} d_{\nu,i}^2.
\]
(5.182)

It should be pointed out that the sine coefficients derived from an "error free" orbit model corresponding to a gravitational field model are rigorous because of the orthogonality properties of the (discrete) sine functions. This means, they fulfill also the Eq. (5.13) etc. and can be considered as "true" reference values for the following considerations.

**Spectral representation of Euler and Bernoulli polynomials:** If the coefficients of the Euler and Bernoulli polynomials are determined, e.g. by an approximation procedure according to Eq. (5.119), then the Fourier coefficients can be determined according to Eqs. (5.99) and (5.100), respectively,
\[
d_{2\nu} = \sum_{j=1}^{J} \frac{2(-1)^{j+1}(2j + 1)!}{(2\nu\pi)^{2j+1}} b_{2j+1} + R_{2\nu},
\]
and
\[
d_{2\nu-1} = \sum_{j=1}^{J} \frac{4(-1)^{j}(2j)!}{(2\nu - 1)^{2j+1} \pi^{2j+1}} e_{2j} + R_{2\nu-1},
\]
with the remainder terms \( R_{2\nu} \) and \( R_{2\nu-1} \). In practical applications, one has to disregard these remainder terms. The coefficients up to index \( J_{max} = 6 \) can be written for even indices \( 2\nu \):
\[
d_{2\nu} \approx 6 \sum_{j=1}^{6} \frac{2(-1)^{j+1}(2j + 1)!}{(2\nu\pi)^{2j+1}} b_{2j+1} = \]
\[
= \frac{3}{2(\nu\pi)^3} b_3 - \frac{15}{2(\nu\pi)^5} b_5 + \frac{315}{4(\nu\pi)^7} b_7 - \frac{2835}{8(\nu\pi)^9} b_9 + \frac{159025}{4(\nu\pi)^{11}} b_{11} - \frac{6081075}{4(\nu\pi)^{13}} b_{13},
\]
(5.183)
and for coefficients with odd indices \( 2\nu - 1 \),
\[
d_{2\nu-1} \approx 6 \sum_{j=1}^{6} \frac{4(-1)^{j}(2j)!}{(2\nu - 1)^{2j+1} \pi^{2j+1}} e_{2j} = \]
\[
= -\frac{8}{(2\nu - 1)^3 \pi^3} e_2 + \frac{96}{(2\nu - 1)^5 \pi^5} e_4 - \frac{2880}{(2\nu - 1)^7 \pi^7} e_6 + 
\]
\[
+ \frac{161280}{(2\nu - 1)^9 \pi^9} e_8 - \frac{14515200}{(2\nu - 1)^{11} \pi^{11}} e_{10} + \frac{1916006400}{(2\nu - 1)^{13} \pi^{13}} e_{12}.
\]
(5.184)

Because of the fact that the remainder terms of the direct Fourier analysis, Eq. (5.95),
\[
R_F(\tau) = \sum_{\nu=n+1}^{\infty} d_{\nu} \sin(\nu \pi \tau),
\]
(5.185)
and those of the spectra of the Euler-Bernoulli polynomials, Eq. (5.101), for \( J_{max} = 6 \)
\[
R_{2\nu} = \sum_{j=7}^{\infty} \frac{2(-1)^{j+1}(2j + 1)!}{(2\nu\pi)^{2j+1}} b_{2j+1} = \beta \frac{2}{(2\nu\pi)^4} \int_{\tau'=0}^{1} d^{[4]}(\tau') \sin 2\nu \pi \tau' d\tau',
\]
(5.186)
and Eq. (5.102), respectively, for $J_{\text{max}}=6$

$$R_{2\nu-1} = \sum_{\nu=1}^{\infty} \frac{4(-1)^{\nu}(2\nu)!}{(2\nu-1)!} c_{2\nu} = \beta \frac{2}{(2\nu-1)14^{14}} \int_{\tau'=0}^{1} d^{[14]}(\tau') \sin(2\nu - 1)\pi\tau' d\tau',$$  \hspace{1cm} (5.187)

are different, we cannot expect identical results as derived from either the force function or by a sine analysis, Eq. (5.185).

Before we discuss the "real orbit" simulations, we will investigate the error of a simplified determination of the coefficients $d_{2\nu}$ and $d_{2\nu-1}$ according to the Eqs. (5.183) and (5.184). As already mentioned in Sec. 5.3.2.2, the coefficients of the Fourier series up to index $n$ can be determined from the coefficients of the Euler-Bernoulli, e.g. up to degree $J=20$ according to Eq. (5.132)

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} I_1^1 & \cdots & I_1^{13} \\ \vdots & \ddots & \vdots \\ I_n^1 & \cdots & I_n^{13} \end{pmatrix} \begin{pmatrix} (E_{\nu,1}^1)^T \\ \vdots \\ (E_{\nu,1}^3)^T \end{pmatrix} \begin{pmatrix} e_{1,2} \\ \vdots \\ e_{1,13} \end{pmatrix},$$

in principle, up to an arbitrarily chosen upper index $n$. The same results can be achieved by determining the coefficients $d_{\nu}$ numerically by a discrete Fourier analysis according to Eq. (5.51). In the following, a 30 arc minute arc has been simulated based on a gravity field model up to a maximal spherical harmonic degree of $N_F=300$. As reference motions, the straight line, the ellipse mode and a dynamical reference arc based on a gravity field up to degree $N_R=30$ has been selected. To these three cases, an Euler-Bernoulli polynomial with $J_{\text{max}}=6$ has been fitted by a least squares adjustment procedure. Then only the Euler-Bernoulli function has been developed in a Fourier series. This has been performed first by the analytical determination according to Eqs. (5.183) and (5.184) and then directly. The direct Fourier analysis has been performed, based on a sampling rate of 10 seconds. For the present example, the interpolation case has been applied, so that $n=K=179$ nodal points (except the boundary values) are available according to Eq. (5.52),

$$d(\tau_k) \quad \text{with} \quad \tau_k = \frac{k}{K+1}, \quad k = 1, 2, 3, \ldots, K, \quad \tau_k \in [0, 1],$$

resulting in 179 amplitudes according to Eq. (5.51),

$$d_{\nu} \approx \frac{2}{K+1} \sum_{k=1}^{K} d(\tau_k) \sin \left( \frac{\nu\pi k}{K+1} \right).$$

The differences for the straight line mode, the ellipse mode and the dynamical reference orbit mode are shown in Figs. 5.29(a), 5.29(b) and 5.29(c). The figures clearly show increasing differences with increasing index $\nu$. The absolute differences depend also on the size of the amplitudes. But it is interesting to note that the relation between the approximated coefficients, derived according to Eqs. (5.183) and (5.184), $d_{\nu,i}(i=1, 2, 3)$ and derived numerically by discrete Fourier analysis, $d_{\nu,i}^*(i=1, 2, 3)$,

$$\frac{d_{\nu,i}}{d_{\nu,i}^*} = f(\nu),$$ \hspace{1cm} (5.188)

follows the same error characteristic $f(\nu)$, independent of the coordinates and the mode of the reference motion and also independent of the coefficient, whether it is odd or even (Figs. 5.30 to 5.35). This result corresponds approximately with the functional dependency of the remainder functions Eqs. (5.186) and (5.187).

The convergence behavior of the Fourier series should be similar to the convergence of the simulated orbit. To demonstrate this, the differences are shown between the amplitudes of the orbit, determined on the one hand by discrete Fourier analysis according to Eq. (5.51) and on the other hand from the discrete Fourier analysis of the Euler-Bernoulli orbit ($J_{\text{max}}=6$). Fig. 5.38 shows the case for the straight line mode and Fig. 5.39 for the ellipse mode. The differences start with zero and become slightly larger with increasing
Figure 5.29: Effects of the remainder terms Eqs. (5.186) and (5.187): Differences of the spectra of an Euler-Bernoulli polynomial of degree $J_{max}=6$, determined by Eqs. (5.183) and (5.184) on the one hand and by discrete Fourier analysis according to Eq. (5.51) on the other hand ((a) for straight line mode (b) for ellipse mode (c) for dynamical reference orbit $N_R=30$ mode).
The residuals in the space domain derived by a Fourier synthesis are shown in Fig. 5.40; the upper index is selected in this case to \( n = 179 \). The residuals are rather large up to 2cm at the boundaries of the arc in accordance with the fact that the amplitudes differ especially in the high frequency part of the spectrum. If the limit of the Fourier series is extended to \( n = 400 \), then the residuals become smaller by one order as shown in Fig. 5.41. The residuals in these figures reflect the remainder term of the Fourier series (Eq. (5.94)). The test computations demonstrate the possibility to derive the Fourier spectrum based on the Euler-Bernoulli polynomial coefficients, in principle, up to an arbitrary degree.

On the other hand the inverse procedure can be used to determine also the coefficients of the Euler and Bernoulli polynomials according to Eq. (5.135) based on the spectrum of the satellite’s arc in terms of directly derived Fourier amplitudes according to Eq. (5.51). This is an over-determined problem, because only \( 3 \cdot 12 = 24 \) coefficients in case of an upper limit of \( J_{\text{max}} = 6 \) have to be determined by \( 3 \cdot 179 \) amplitudes. The results are shown in the Tables 5.7 and 5.8 together with the (true) polynomial coefficients derived by a numerically very precise technique for the straight line and ellipse modes as explained in Sec. 5.4.2.1. If the Euler-Bernoulli polynomials are used to determine the ephemerides of the arc, then the result will slightly differ from the original positions as shown in Figs. 5.43 and 5.45 for the straight line and an ellipse modes, respectively.

**Figure 5.30:** Error characteristic \( f(\nu) \) of odd amplitudes (reference motion: straight line mode, \( J_{\text{max}}=6, N_F=300 \)).

**Figure 5.31:** Error characteristic \( f(\nu) \) of even amplitudes (reference motion: straight line mode, \( J_{\text{max}}=6, N_F=300 \)).

**Figure 5.32:** Error characteristic \( f(\nu) \) of odd amplitudes (reference motion: ellipse mode, \( J_{\text{max}}=6, N_F=300 \)).

**Figure 5.33:** Error characteristic \( f(\nu) \) of even amplitudes (reference motion: ellipse mode, \( J_{\text{max}}=6, N_F=300 \)).
5.4. Numerical Verifications

Figure 5.34: Error characteristic $f(\nu)$ of odd amplitudes (reference motion: dynamical reference orbit mode $N_R=30, J_{\text{max}}=6, N_F=300$).

Figure 5.35: Error characteristic $f(\nu)$ of even amplitudes (reference motion: reference orbit $N_R=30$ mode, $J_{\text{max}}=6, N_F=300$).

Figure 5.36: Differences between the orbit for a gravitational field complete up to degree 2 order 0 and the approximations by a series in terms of Euler-Bernoulli polynomial $d_p^6(\tau)$ (reference motion: ellipse mode).

Figure 5.37: Differences between the orbit for a gravitational field complete up to degree and order 20 and the approximations by a series in terms of Euler-Bernoulli polynomial $d_p^6(\tau)$ (reference motion: ellipse mode).

Table 5.7: Coefficients of the Euler-Bernoulli polynomials either determined by a least squares fit based on the (error free) ephemerides and derived from the sine coefficient according to Eq. (5.135) (reference motion: straight line mode, $J_{\text{max}}=6, N_F=300$).

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<tr>
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<td>0.917094E+07</td>
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<tr>
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<td>0.220624E+07</td>
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<td>0.417257E+07</td>
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Figure 5.38: Differences between the amplitudes of the orbit, determined on the one hand by direct Fourier analysis according to Eq. (5.51) and on the other hand from the discrete Fourier analysis of the Euler-Bernoulli orbit (reference motion: straight line mode, $J_{\text{max}} = 6$, gravitational field degree $N_F = 300$).

Figure 5.39: Differences between the amplitudes of the orbit, determined on the one hand by direct Fourier analysis according to Eq. (5.51) and on the other hand from the discrete Fourier analysis of the Euler-Bernoulli orbit (reference motion: ellipse mode, $J_{\text{max}} = 6$, gravitational field degree $N_F = 300$).

Table 5.8: Coefficients of the Euler-Bernoulli polynomials either determined by a least squares fit based on the (error free) ephemerides and derived from the sine coefficient according to Eq. (5.135) (reference motion: ellipse mode, $J_{\text{max}} = 6$, $N_F = 300$).

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<tr>
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<th>$e_{(2j)}$ (derived)</th>
<th>$b_{(2j+1)}$ (derived)</th>
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### 5.4. Numerical Verifications

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#### Figure 5.40: Residuals of the Fourier series with an upper summation limit of $n=179$ in space domain based on Eq. (5.132) with a maximal Euler-Bernoulli index $J_{\text{max}} = 6$ (reference motion: *ellipse* mode).

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#### Figure 5.41: Residuals of the Fourier series with an upper summation limit of $n=400$ in space domain based on Eq. (5.132) with a maximal Euler-Bernoulli index $J_{\text{max}} = 6$ (reference motion: *ellipse* mode).

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#### Figure 5.42: Residuals based on the least squares fit: observed orbit-E.B. polynomials (reference motion: *straight line* mode, $J_{\text{max}}=6$, $N_F=300$).

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<tr>
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#### Figure 5.43: Residuals: observed orbit-E.B. polynomials based on Eq. (5.135) (reference motion: *straight line* mode, $J_{\text{max}}=6$, $N_F=300$).
Table 5.9: Coefficients of the Euler-Bernoulli polynomials either determined by a least squares fit based on the (error free) ephemerides and derived from the sine coefficient according to Eq. (5.135) (reference motion: dynamical reference orbit $N_R=30$, $J_{max}=6$, $N_F=300$).

<table>
<thead>
<tr>
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<td>6</td>
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<td>0.189671E+05</td>
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Figure 5.44: Residuals based on the least squares fit: observed orbit-E.B. polynomials (reference motion: ellipse mode, $J_{max}=6$, $N_F=300$).

Figure 5.45: Residuals: observed orbit-E.B. polynomial based on Eq. (5.135) (reference motion: ellipse mode, $J_{max}=6$, $N_F=300$).

Figure 5.46: Position residuals of the ephemerides derived from the Euler-Bernoulli polynomials based on the least squares fit (reference motion: dynamical reference orbit $N_R=30$, $J_{max}=6$, $N_F=300$).

Figure 5.47: Position residuals of the ephemerides derived from the Euler-Bernoulli polynomials based on Eq. (5.135) (reference motion: dynamical reference orbit $N_R=30$, $J_{max}=6$, $N_F=300$).
6. Integrated Kinematic-Dynamic Orbit Determination

6.1 Kinematical Orbit Determination

In the preceding sections, the theoretical foundation of the integrated kinematical-dynamical orbit determination procedure is presented. Some additional more application oriented aspects but important as pre-requisite for the following developments are discussed as well. The proposed orbit determination approach is characterized by the fact that the semi-analytical representation of the satellite’s arc can be considered as solution to Newton’s equation of motion on the one hand or by an empirical approximation of the satellite’s arc on the other hand. In the former case, the boundary vectors are computed by a least squares fitting process to the (pseudo) observations while the orbit parameters have to fulfill the restrictions caused by the force function acting on the satellite. In the latter case, the orbit parameters are determined together with the boundary vectors of the specific arc by an adjustment process, where the parameters are determined such that the square sum of the residuals is minimized. As outlined in detail, the satellite’s arc can be represented by a sine series on the one hand and by a combination of Euler and Bernoulli polynomials on the other hand. Preferable is a combinations between both representations, which allow to take advantage by the specific features of these two semi-analytical orbit representations. There are various modes of the orbit representation and different alternatives of a kinematical orbit determination, which shall be discussed in the following. First of all, an overview is given which repeats some essential facts from the preceding sections of this thesis.

The satellite’s arc is represented by the function:

\[ r(\tau) = \bar{r}(\tau) + d(\tau), \]

where the reference motion can be modeled by \( \bar{r}(\tau) \) either according to Eq. (5.10),

\[ \bar{r}(\tau) = (1 - \tau) r_A + \tau r_B, \]

or according to Eq. (5.20),

\[ \bar{r}(\tau) = \frac{\sin \mu(1 - \tau)}{\sin \mu} r_A + \frac{\sin \mu \tau}{\sin \mu} r_B, \]

or according to Eq. (5.37),

\[ \bar{r}(\tau) = \bar{r}(\tau) + \bar{x}(\tau). \]

The difference function \( d(\tau) \) can be formulated as solution to the respective boundary value problem as treated in Sec. 5.2, based on Eqs. (5.9), (5.19) or (5.32) and interpreted as Fourier series or sine series, respectively, according to Eq. (5.39),

\[ d(\tau) \equiv d^F(\tau) = \sum_{\nu=1}^{\infty} d_\nu \sin(\nu \pi \tau) \approx \sum_{\nu=1}^{n} d_\nu \sin(\nu \pi \tau) = d^n_F(\tau). \]

This series can be used as pure kinematical orbit representation with the kinematical orbit parameters \( d_\nu \) as treated in detail in Sec. 6.1.1.

In Sec. 5.3, we transformed this Fourier series into a series of Euler and Bernoulli polynomials,

\[ d(\tau) \equiv d^F(\tau) = d^P(\tau), \]
with
\[ d(\tau) \equiv d_P^F(\tau) = \sum_{j=1}^{\infty} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{\infty} b_{2j+1} B_{2j+1}(\tau) \approx \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau) = d_P^J(\tau). \]

Here, the coefficients of the Euler and Bernoulli polynomials, \( e_{2j} \) and \( b_{2j+1} \), can be used as kinematical parameters of the orbit representation but they can be related also to the force function acting on the satellite. The different possibilities are discussed in Sec. 6.1.2.

Besides these two principle possibilities, another hybrid or combined version is possible where some specific advantageous features of these two representations can be exploited. The function \( d(\tau) \) and its corresponding series representation is split into a modified Fourier series \( d_P^F(\tau) \) and a series \( d_P^J(\tau) \) in terms of Euler- and Bernoulli polynomials, Eq. (5.90), with properly selected upper indices \( n \) and \( J \),

\[ d(\tau) \equiv d_P^F(\tau) + d_P^J(\tau). \]

There are various possibilities to combine these two solution series as will be shown in Sec. 6.1.3.

If \( K \) positions at the discrete epochs \( \tau_1, \cdots, \tau_K \) are available, derived from GNSS observations as treated in chapter 4, then we can use the following matrix notation,

\[ r = \bar{r} + d, \]

with the ephemerides of satellite positions at the observed epochs,

\[ r := \begin{pmatrix} r(\tau_1) \\ \vdots \\ r(\tau_K) \end{pmatrix}, \]

the contribution of the reference motion according to Eqs. (5.10), (5.20), or (5.37)

\[ \bar{r} := \begin{pmatrix} \bar{r}(\tau_1) \\ \vdots \\ \bar{r}(\tau_K) \end{pmatrix} = \begin{pmatrix} a(\tau_1) \\ \vdots \\ a(\tau_K) \end{pmatrix} r_A + \begin{pmatrix} b(\tau_1) \\ \vdots \\ b(\tau_K) \end{pmatrix} r_B = a r_A + b r_B, \]

and the linear factors \( a \) and \( b \) according to the reference motion terms and the discrete constituents of the difference function,

\[ a := \begin{pmatrix} a(\tau_1) \\ \vdots \\ a(\tau_K) \end{pmatrix}, \quad b := \begin{pmatrix} b(\tau_1) \\ \vdots \\ b(\tau_K) \end{pmatrix}, \quad d := \begin{pmatrix} d(\tau_1) \\ \vdots \\ d(\tau_K) \end{pmatrix}. \]

The latter matrix can be represented either by the sine coefficients or by the Euler and Bernoulli coefficients or by a combination of them, as discussed in the following.

### 6.1.1 Fourier Series

The difference function \( d(\tau) \) - in principle an infinite series - is approximated by a sine series with an upper limit \( n \), \( d_P^F(\tau) \). The 3\( n \) series coefficients \( d_n \), together with the 6 coordinates of the boundary vectors \( r_A \) and \( r_B \) are the parameters, which have to be determined, based on the observations of the satellite’s motion. These observations can be provided as ephemerides of positions as treated in chapter 5 or as direct GNSS observables as treated in chapter 4. The foundation of the observation equations are presented in chapter 5. If this is done without any additional dynamic information, then a continuous approximation function is available which is of pure kinematical character, because any further derivative with respect to the time can be determined as shown, at least theoretically, in Sec. 5.3.3 in a straightforward way.

The determination of the kinematical orbit parameters can be determined from positions, derived in a first preparation step by a geometrical orbit determination procedure (see chapter 4), or directly by the carrier phase GPS-SST observations, together with observation specific corrections. Both procedures are presented in the following.
6.1. Kinematical Orbit Determination

6.1.1.1 Position Observations

The orbit of a satellite can be represented in a semi-analytical way by one of the Eqs. (5.9), (5.19) or (5.36),

$$r(\tau) = \bar{r}(\tau) + \sum_{\nu=1}^{\infty} c_\nu \sin(\nu \pi \tau) = \bar{r}(\tau) + d(\tau) = \bar{r}(\tau) + d_\infty F(\tau).$$  \(6.11\)

If the upper index \(\nu\) is limited by a sufficient high number \(n\), then the difference function reads,

$$d(\tau) \equiv d_\infty F(\tau) = \sum_{\nu=1}^{n} d_\nu \sin(\nu \pi \tau) \approx n \sum_{\nu=1}^{n} d_\nu \sin(\nu \pi \tau) = d_n F(\tau),$$  \(6.12\)

and the LEO orbit can be represented as,

$$r(\tau) = \bar{r}(\tau) + \sum_{\nu=1}^{n} d_\nu \sin(\nu \pi \tau).$$  \(6.13\)

This formula can be considered as a linear approximation of the satellite’s arc. If the absolute position of the LEO at the epoch \(\tau\) is derived geometrically (see chapter 4) with its variance-covariance matrix, then the linear observation equation reads,

$$r(\tau) = \bar{r}(\tau) + \sum_{\nu=1}^{n} d_\nu \sin(\nu \pi \tau), \quad C(\tau) = \begin{pmatrix} \sigma_{x_x}^2 & \sigma_{x_y} \sigma_{y_x} & \sigma_{x_z} \sigma_{z_x} \\ \sigma_{y_x} \sigma_{x_y} & \sigma_{y_y}^2 & \sigma_{y_z} \sigma_{z_y} \\ \sigma_{z_x} \sigma_{x_z} \sigma_{z_y} \sigma_{y_z} & & \sigma_{z_z}^2 \end{pmatrix}. \quad \text{(Sym.)}$$  \(6.14\)

If the LEO positions are observed at the epochs \(\tau_1, \ldots, \tau_K\), then the matrix of observation equations reads together with the a-priori variance-covariance matrix \(C_r\),

$$\begin{pmatrix} r(\tau_1) \\ \vdots \\ r(\tau_K) \end{pmatrix} = \begin{pmatrix} C(\tau_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C(\tau_K) \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = S^T d_n,$$  \(6.15\)

with the contribution of the reference motion \(\bar{r}\), derived by an expression of the form of Eqs. (5.10), (5.20) or (5.37), and the discrete constituents of the difference function,

$$d = \begin{pmatrix} d(\tau_1) \\ \vdots \\ d(\tau_K) \end{pmatrix} = \begin{pmatrix} \sin(\pi \tau_1) & \cdots & \sin(n \pi \tau_1) \\ \vdots & \ddots & \vdots \\ \sin(\pi \tau_K) & \cdots & \sin(n \pi \tau_K) \end{pmatrix} \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = S^T d_n,$$  \(6.16\)

so that it reads with Eqs. (6.9) and (6.10),

$$r = a r(\tau_A) + b r(\tau_B) + S^T d_n, \quad C_r.$$  \(6.17\)

Gauss-Markov model for observed positions: In a pure kinematical orbit determination application, the boundary vectors \(r_A\) and \(r_B\) as well as the Fourier coefficients \(d_\nu\) are the unknown parameters so that the observation model reads,

$$\begin{pmatrix} r(\tau_1) \\ \vdots \\ r(\tau_K) \end{pmatrix} = \begin{pmatrix} a & b & S^T \end{pmatrix} \begin{pmatrix} r_A \\ r_B \\ d_1 \\ \vdots \\ d_n \end{pmatrix}, \quad C_r.$$  \(6.18\)
or reformulated with the observation vector, the design matrix and the unknown parameters as,

\[
I = \begin{pmatrix}
  r(\tau_1) \\
  \vdots \\
  r(\tau_K)
\end{pmatrix}, \quad A = \begin{pmatrix}
  a & b & S^T
\end{pmatrix}, \quad x = \begin{pmatrix}
  r_A \\
  r_B \\
  d_1 \\
  \vdots \\
  d_n
\end{pmatrix},
\]

or, respectively, with the dimensions \( l = 3K \) and \( n = 3n + 6 \)

\[
I_i = A_{ij}x_i, \quad C_r.
\]

The unknown parameters and its a-posteriori variance-covariance matrix can be estimated directly as usual

\[
\hat{x} = (A^TC_r^{-1}A)^{-1}A^TC_r^{-1}I, \quad C_{\hat{x}} = (A^TC_r^{-1}A)^{-1}.
\]

Either in case of a kinematical orbit determination or in case of the dynamical orbit determination concept as discussed later, the upper index \( n \) has to be selected sufficiently high to avoid Gibbs’ effects especially at the boundaries of the satellite’s arcs. In Sec. 5.4.1.1, the approximation characteristics of the Fourier series based on different reference motions (straight line, ellipse, reference orbit) are discussed for the error free case. Usually, the number of observed three dimensional positions very often is not sufficient to provide a safe redundancy in the least squares adjustment of the orbit determination process. This is the reason that a modification to this approach is necessary.

### 6.1.1.2 SST Carrier Phase Observations

The Fourier amplitudes as well as the boundary positions of the satellite’s arc can be directly estimated from the GPS-SST observations. The carrier phase observations at the frequency \( i \) between the GPS satellite \( s \) and the GPS receiver \( r \) on-board LEO at the normalized epoch \( \tau \) can be formulated as (refer to chapter 4 for details),

\[
\Phi_{r,i}^s(\tau) = \|r_{m}^s(\tau) - r(\tau)\| + c\delta t_r(\tau) + \lambda_i A_{\tau,i}^s + d_{M,\Phi}(\tau) + \epsilon_{\tau,\Phi}(\tau) + \epsilon_{\tau,\Phi}^r.
\]

If the LEO orbit representation Eq. (6.13) is inserted in the Eq. (6.22), then the carrier phase observation equation reads as follows

\[
\Phi_{r,i}^s(\tau) = \|r_{m}^s(\tau) - r(\tau) - \sum_{\nu=1}^{n}d_{\nu}\sin(\nu\pi\tau)\| + c\delta t_r(\tau) + \lambda_i A_{\tau,i}^s + d_{M,\Phi}(\tau) + \epsilon_{\tau,\Phi}(\tau) + \epsilon_{\tau,\Phi}^r.
\]

The GPS-SST carrier phase observation equations are linear with respect to the LEO clock offsets and the GPS ambiguity parameters but are non-linear with respect to the LEO orbit representation parameters, namely the LEO boundary positions and the Fourier amplitudes. The carrier phase observation equation can be linearized with respect to the LEO orbit representation parameters at frequency \( i \) as,

\[
\Phi_{r,i}^s(\tau) = \Phi_{r,i,0}^s(\tau) + \frac{\partial\Phi_{r,i}^s(\tau)}{\partial x}\Big\|_{x=x_0} (x - x_0),
\]

with

\[
x := \begin{pmatrix}
  x_r \\
  x_i \\
  x_A
\end{pmatrix}, \quad x_r := \begin{pmatrix}
  r_A \\
  r_B \\
  d_1 \\
  \vdots \\
  d_n
\end{pmatrix}, \quad x_i := \begin{pmatrix}
  c\delta t_r(\tau_1) \\
  \vdots \\
  c\delta t_r(\tau_K)
\end{pmatrix}, \quad x_A := \begin{pmatrix}
  \lambda_i A_{\tau,i}^s \\
  \vdots \\
  \lambda_i A_{\tau,i}^{s_{\text{max}}}
\end{pmatrix}.
\]
contains the LEO boundary positions, the Fourier amplitudes up to the Fourier index $n$, the LEO clock offsets at every of the $K$ observations and the GPS ambiguity parameters for $m$ tracked GPS satellites. If the ionosphere-free carrier phase observations (index $i = 3$) are used in the procedure (see Sec. 2.7.3), then the multiplication of the design matrix by the corrections to the approximations of the unknowns reads

$$
\frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{x}} (\mathbf{x} - \mathbf{x}_0) = \begin{pmatrix} \frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{x}_r} & \frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{x}_t} & \frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{x}_A} \end{pmatrix} \begin{pmatrix} \mathbf{x}_r - \mathbf{x}_{r,0} \\ \mathbf{x}_t - \mathbf{x}_{t,0} \\ \mathbf{x}_A - \mathbf{x}_{A,0} \end{pmatrix}.
$$

The partial derivatives of the ionosphere-free carrier phase GPS-SST observation $\Phi_{s,3}^r$ with respect to the non-linear parameters can be determined by applying the chain rule.

Partial derivatives of the carrier phase observations with respect to the LEO boundary positions and Fourier amplitudes: The partial derivatives of the ionosphere-free carrier phase GPS-SST observations with respect to the LEO boundary positions and the Fourier amplitudes, $\mathbf{x}_r$, can be written as follows,

$$
\frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{x}_r} = \frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{r}(\tau)} \frac{\partial \mathbf{r}(\tau)}{\partial \mathbf{x}_r},
$$

or in matrix notation,

$$
\mathbf{A}_{\Phi_x}^s(\tau) = \mathbf{a}_{\Phi_x}^s(\tau) \mathbf{A}_{\mathbf{x}_r}^s(\tau).
$$

according to Eqs. (4.25) and (4.31),

$$
\mathbf{a}_{\Phi_x}^s(\tau) = \begin{pmatrix} e_{x,3}^s(\tau) & e_{y,3}^s(\tau) & e_{z,3}^s(\tau) \end{pmatrix},
$$

$$
e_{x,3}^s(\tau) = \frac{x_r(\tau) - x^s(\tau - \tau_s^s)}{\rho^s_r(\tau)},
$$

$$
e_{y,3}^s(\tau) = \frac{y_r(\tau) - y^s(\tau - \tau_s^s)}{\rho^s_r(\tau)},
$$

$$
e_{z,3}^s(\tau) = \frac{z_r(\tau) - z^s(\tau - \tau_s^s)}{\rho^s_r(\tau)},
$$

with $\tau_s^s$ as the GPS signal travel time between the GPS satellite $s$ and the GPS receiver $r$ on-board LEO and with the factors $a(\tau)$ and $b(\tau)$ according to Eq. (6.9) and the sine coefficients in $\mathbf{H}^T(\tau)$ according to Eq. (6.16),

$$
\mathbf{A}_{\mathbf{x}_r}^s(\tau) = \begin{pmatrix} a(\tau) \mathbf{I} & b(\tau) \mathbf{I} & \mathbf{H}^T(\tau) \end{pmatrix},
$$

$$\mathbf{H}^T(\tau) = \begin{pmatrix} \sin \pi \tau & \ldots & \sin n \pi \tau \end{pmatrix},
$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Partial derivatives of the carrier phase observation with respect to the LEO clock offset: A carrier phase GPS-SST observation is linear with respect to the LEO clock offset. The partial derivatives of the ionosphere-free carrier phase observation with respect to the LEO clock offset reads for an observation epoch $\tau$,

$$
\frac{\partial \Phi_{s,3}^r(\tau)}{\partial \mathbf{x}_t} = \mathbf{a}_{\Phi_x}^s(\tau),
$$

with

$$
\mathbf{a}_{\Phi_x}^s(\tau) = \begin{pmatrix} 0 & \ldots & 1 & \ldots & 0 \end{pmatrix}.
$$
Partial derivatives of the carrier phase observation with respect to the GPS ambiguity terms:

A carrier phase GPS-SST observation is linear with respect to the GPS ambiguity term. Therefore, the partial derivative of the ionosphere-free carrier phase observation equation with respect to the ambiguity term of GPS satellite \( s \) at the observed epoch reads,

\[
\frac{\partial \Phi^s_{r,3}(\tau)}{\partial \mathbf{x}_A} = \mathbf{a}^s_{\mathbf{x}_A}(\tau),
\]

with

\[
\mathbf{a}^s_{\mathbf{x}_A}(\tau) = \begin{pmatrix} 0 & \ldots & 1 & \ldots & 0 \end{pmatrix}.
\]

Gauss-Markov model for SST carrier phase observations: A pure kinematical orbit determination from SST carrier phase observations requires the determination of the kinematical orbit parameters and additional observation specific corrections. The ionosphere-free observation equation for the GPS satellite \( s \) and the GPS receiver \( r \) on-board LEO reads for an epoch \( \tau \) as follows,

\[
\Delta \Phi^s_{r,3}(\tau) = \mathbf{a}^s_1(\tau) \mathbf{A}^s_{x_1}(\tau) (\mathbf{x}_r - \mathbf{x}_{r,0}) + \mathbf{a}^s_2(\tau) (\mathbf{x}_i - \mathbf{x}_{i,0}) + \mathbf{a}^s_3(\tau) (\mathbf{x}_A - \mathbf{x}_{A,0}),
\]

\[
w^s_{r,\Phi_3}(\tau) = \frac{\sigma_0^2}{\sigma_{\Phi_3}(\tau)^2} \cos^2 \left( \frac{\pi}{2} \sigma_{\Phi_3}(\tau) \right),
\]

with the initial standard deviation \( \sigma_0 \), the standard deviation of the ionosphere-free carrier phase observation \( \sigma_{\Phi_3}(\tau) \) and the zenith distance \( z_r(\tau) \) from the GPS receiver \( r \) on-board LEO to the GPS satellite \( s \) at time \( \tau \). \( w^s_{r,\Phi_3}(\tau) \) is the weight of the carrier phase observation \( \Phi^s_{r,3}(\tau) \).

If \( m_r \) is the total number of carrier phase observations at time \( \tau \) from the GPS satellites \( s_1, \ldots, s_{m_r} \) to the GPS receiver \( r \) on-board LEO, the Gauss-Markov model for the three unknown groups can be summarized as follows,

\[
\begin{pmatrix}
\Delta \Phi^s_{r,3}(\tau) \\
\vdots \\
\Delta \Phi^{s_{m_r}}_{r,3}(\tau)
\end{pmatrix} =
\begin{pmatrix}
\mathbf{a}^s_1(\tau) \mathbf{A}^s_{x_1}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r}}(\tau) \mathbf{A}^{s_{m_r}}_{x_1}(\tau)
\end{pmatrix}
\begin{pmatrix}
(\mathbf{x}_r - \mathbf{x}_{r,0}) \\
\vdots \\
(\mathbf{x}_{i,0} - \mathbf{x}_{i,0}) \\
\vdots \\
(\mathbf{x}_A - \mathbf{x}_{A,0})
\end{pmatrix},
\]

or in matrix notation at the epoch \( \tau \) as,

\[
\Delta \mathbf{l}(\tau) = \mathbf{A}(\tau) \Delta \mathbf{x},
\]

with the weight matrix of the carrier phase observation at the epoch \( \tau \) (inverse of the variance-covariance matrix of the carrier phase observations, see Sec. 3.3),

\[
\mathbf{W}(\tau) =
\begin{pmatrix}
w^{s_1}_{r,\Phi_3}(\tau) & 0 & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & w^{s_{m_r}}_{r,\Phi_3}(\tau) & 0 \\
0 & \ldots & 0 & w^{s_{m_r}}_{r,\Phi_3}(\tau)
\end{pmatrix}.
\]

If \( l \) and \( u \) (\( l \geq u \)) are the total numbers of the ionosphere-free carrier phase GPS-SST observations at epochs \( \tau_1, \ldots, \tau_K \) and the unknown parameters, respectively, then the Gauss-Markov model for all tracked GPS satellites and all observation epochs reads

\[
\begin{pmatrix}
\Delta \mathbf{l}(\tau_1) \\
\vdots \\
\Delta \mathbf{l}(\tau_K)
\end{pmatrix} =
\begin{pmatrix}
\mathbf{A}(\tau_1) \\
\vdots \\
\mathbf{A}(\tau_K)
\end{pmatrix} \Delta \mathbf{x},
\]
or in matrix notation as,

$$\Delta \hat{x}_{(i)} = A_{(i \times u)} \Delta x_{(u)}, \quad W_l = \begin{pmatrix} W(\tau_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W(\tau_K) \end{pmatrix},$$  \hspace{1cm} (6.39)$$

with

- \( l \) the total number of carrier phase GPS-SST observations at epochs \( \tau_1, \ldots, \tau_K \),
- \( u \) the total number of unknowns as \( u = u_r + u_t + u_A \),
- \( u_r \) the number of LEO boundary position coordinates and the number of Fourier amplitudes for an upper index \( n \) (\( u_r = 6 + 3n \)),
- \( u_t \) the total number of the LEO clock offsets except the first epoch (\( u_t = K - 1 \)),
- \( u_A \) the GPS ambiguity parameters.

The corrections to the approximate unknown parameters and its a-posteriori variance-covariance matrix can be determined according to

$$\Delta \hat{x} = (A^T W_l A)^{-1} A^T W_l \Delta l, \quad C_{\Delta \hat{x}} = (A^T W_l A)^{-1}. \hspace{1cm} (6.40)$$

The estimated unknown parameters in Eq. (6.40), \( \Delta \hat{x} \), are the corrections to the initial LEO boundary positions, the Fourier amplitudes, the initial LEO clock offsets and the initial ambiguities of all observed GPS satellites from the GPS receiver on-board LEO. Because of the non-linearized observation model, an iteration procedure is necessary. The convergence of the unknowns can be achieved after a few iterations,

$$\hat{x}_{(i)} = \hat{x}_{(i-1)} + \Delta \hat{x}_{(i)} = \hat{x}_{(i-1)} + \left( A^T_{(i)} W_l A_{(i)} \right)^{-1} A^T_{(i)} W_l \Delta l_{(i)}.$$  \hspace{1cm} (6.41)

### 6.1.2 Euler- and Bernoulli Polynomials

As shown in Eq. (5.74), the sine series can be transformed in a series of Euler- and Bernoulli polynomials. In case of an infinite series it holds

$$d(\tau) := r(\tau) - \bar{r}(\tau) = \sum_{j=1}^{\infty} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{\infty} b_{2j+1} B_{2j+1}(\tau) = d^\infty_{\bar{r}}(\tau). \hspace{1cm} (6.42)$$

If the upper index \( j \) is limited by a sufficient high number \( J \), then the difference function reads,

$$d(\tau) \equiv d^J_{\bar{r}}(\tau) = \sum_{j=1}^{\infty} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{\infty} b_{2j+1} B_{2j+1}(\tau) \approx \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau) = d^J_{\bar{r}}(\tau). \hspace{1cm} (6.43)$$

This formula can be considered as a linear approximation of the satellite’s arc. In this case, the \( 6J \) coefficients \( e_{2j} \) and \( b_{2j+1}, \ j = 1, \ldots, J \), of the Euler- and Bernoulli polynomials together with the 6 coordinates of the boundary vectors \( r_A \) and \( r_B \) are the parameters to be determined based on the observations. In the following, again both cases of observation types for the kinematical orbit determination procedure are outlined.

#### 6.1.2.1 Position Observations

If the observations are represented by absolute positions derived according to the geometrical orbit determination procedure as outlined in chapter 4, then the observation model reads

$$r(\tau) = \bar{r}(\tau) + \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau),$$  \hspace{1cm} (6.44)
with the variance-covariance matrix for the position coordinates

$$
C(\tau) = \begin{pmatrix}
\sigma^2_{x,\tau} & \sigma^2_{x,y,\tau} & \sigma^2_{x,z,\tau} \\
\sigma^2_{y,x,\tau} & \sigma^2_{y,\tau} & \sigma^2_{y,z,\tau} \\
\sigma^2_{z,x,\tau} & \sigma^2_{z,y,\tau} & \sigma^2_{z,\tau}
\end{pmatrix}_{\text{Sym.}}.
$$  \hspace{1cm} (6.45)

The observation model for all observations at the epochs \(\tau_1, \ldots, \tau_K\) reads in matrix notation,

$$
r = r + d, \quad C_r = \begin{pmatrix}
C(\tau_1) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & C(\tau_K)
\end{pmatrix},
$$  \hspace{1cm} (6.46)

with the contribution of the reference motion \(\vec{r}\), derived by an expression of the form of Eqs. (5.10), (5.20) or (5.37), and the discrete constituents of the difference function,

$$
d := \begin{pmatrix}
d(\tau_1) \\
\vdots \\
d(\tau_K)
\end{pmatrix} = T \begin{pmatrix}
E^2_{\tau,J+1} \mathcal{I} \\
B^2_{\tau,J+1} \mathcal{I}
\end{pmatrix} \begin{pmatrix}
\hat{e}_{2J} \\
b_{2J+1}
\end{pmatrix},
$$  \hspace{1cm} (6.47)

with the matrix,

$$
T = \begin{pmatrix}
t_1 & \cdots & t_K \\
\vdots & \ddots & \vdots \\
t_1^{2J+1} & \cdots & t_K^{2J+1}
\end{pmatrix}.
$$  \hspace{1cm} (6.48)

so that it reads with Eqs. (6.8) and (6.9),

$$
r = a r(\tau_A) + b r(\tau_B) + T \begin{pmatrix}
E^2_{\tau,J+1} \mathcal{I} \\
B^2_{\tau,J+1} \mathcal{I}
\end{pmatrix} \begin{pmatrix}
\hat{e}_{2J} \\
b_{2J+1}
\end{pmatrix}.
$$  \hspace{1cm} (6.49)

**Gauss-Markov model for observed positions:** If the LEO absolute positions are geometrically determined at the epochs \(\tau_1, \ldots, \tau_K\), then the Gauss-Markov model reads in matrix notation,

$$
\begin{pmatrix}
r(\tau_1) \\
\vdots \\
r(\tau_K)
\end{pmatrix} = \begin{pmatrix}
\mathbf{a} & \mathbf{b} & T \begin{pmatrix}
E^2_{\tau,J+1} \mathcal{I} \\
B^2_{\tau,J+1} \mathcal{I}
\end{pmatrix} & \begin{pmatrix}
r_A \\
r_B \\
\hat{e}_{2J} \\
b_{2J+1}
\end{pmatrix}
\end{pmatrix} C_r.
$$  \hspace{1cm} (6.50)

The observation model reads,

$$
I_{3K} = A_{3K \times (6J+6)} x_{(6J+6)}, \quad C_r,
$$  \hspace{1cm} (6.51)

with

$$
I = \begin{pmatrix}
r(\tau_1) \\
\vdots \\
r(\tau_K)
\end{pmatrix}, \quad A = \begin{pmatrix}
\mathbf{a} & \mathbf{b} & T \begin{pmatrix}
E^2_{\tau,J+1} \mathcal{I} \\
B^2_{\tau,J+1} \mathcal{I}
\end{pmatrix}
\end{pmatrix}, \quad x = \begin{pmatrix}
r_A \\
r_B \\
\hat{e}_{2J} \\
b_{2J+1}
\end{pmatrix}.
$$  \hspace{1cm} (6.52)

The LEO boundary positions and the Euler-Bernoulli polynomials coefficients can be directly estimated as usual by,

$$
\hat{x} = (A^T C_r^{-1} A)^{-1} A^T C_r^{-1} I, \quad C_{\hat{x}} = (A^T C_r^{-1} A)^{-1}.
$$  \hspace{1cm} (6.53)

This system of linear observation equation can serve for a pure kinematical orbit determination, if the coefficients \(\hat{e}_{2J}\) of the Euler polynomials and the coefficients of the Bernoulli polynomials \(b_{2J+1}\) are derived
from the observations by a least squares adjustment process without any dynamical information, together with the boundary values

\[ \mathbf{r}_A := \mathbf{r}(t_A), \quad \mathbf{r}_B := \mathbf{r}(t_B), \quad t_A < t_B, \quad (6.54) \]

demonstrates that an increase of 15 in the size of about \( \text{cm} \) (corresponding to a maximal degree 11 of the Euler and Bernoulli polynomials) the remainder function is reduced in the case of a dynamical reference orbit as reference motion down to about \( \text{cm} \). Additional computations demonstrate that an increase of \( J \) does not improve the approximation quality.

### 6.1.2.2 SST Carrier Phase Observations

The Euler-Bernoulli coefficients can be derived again directly from the carrier phase GPS-SST observations between the GPS satellites and the LEO. The carrier phase observations between the GPS satellite \( s \) and the GPS receiver \( r \) on-board LEO at frequency \( f \) (Eq. (4.27)) at the normalized epoch \( \tau \) can be written as

\[ \Phi^s_{r,i}(\tau) = \| \mathbf{r}_m^{s,1}(\tau) - \mathbf{r}(\tau) \| + c \delta t_r(\tau) + \lambda_i A_r^{s,1} + d_M \Phi_i(\tau) + e_r^{s,\Phi_i}(\tau). \]

If the LEO orbit representation Eq. (6.44) with the approximation Eq. (6.43) is inserted in Eq. (6.22), then the carrier phase SST observation equation reads

\[ \Phi^s_{r,i}(\tau) = \| \mathbf{r}_m^{s,1}(\tau) - \mathbf{r}(\tau) - \sum_{j=1}^J \mathbf{e}_{2j} E_{2j}(\tau) - \sum_{j=1}^J \mathbf{b}_{2j+1} B_{2j+1}(\tau) \| + c \delta t_r(\tau) + \lambda_i A_r^{s,1} + d_M \Phi_i(\tau) + e_r^{s,\Phi_i}(\tau). \quad (6.55) \]

The carrier phase GPS-SST observations are non-linear with respect to the LEO boundary positions and the unknown Euler-Bernoulli coefficients, but linear with respect to the GPS ambiguity terms and the LEO clock offsets. The linear observation equation (Eq. (6.55)) reads as follows,

\[ \Phi^s_{r,i}(\tau) = \Phi^s_{r,i,0}(\tau) + \left. \frac{\partial \Phi^s_{r,i}(\tau)}{\partial \mathbf{x}} \right|_{\mathbf{x} = \mathbf{x}_0} (\mathbf{x} - \mathbf{x}_0), \quad (6.56) \]

with

\[ \mathbf{x} := \begin{pmatrix} \mathbf{x}_{eb} \\ \mathbf{x}_t \\ \mathbf{x}_A \end{pmatrix}, \quad \mathbf{x}_{eb} = \begin{pmatrix} \mathbf{r}_A \\ \mathbf{r}_B \\ \mathbf{r}_{e_{2J}} \\ \mathbf{b}_{2J+1} \end{pmatrix}, \quad \mathbf{x}_t := \begin{pmatrix} c \delta t_r(\tau_1) \\ \vdots \\ c \delta t_r(\tau_K) \end{pmatrix}, \quad \mathbf{x}_A := \begin{pmatrix} \lambda_i A_r^{s,1} \\ \vdots \\ \lambda_i A_r^{s,m} \end{pmatrix}. \quad (6.57) \]

\( \mathbf{x} \) contains the LEO boundary positions, the Euler-Bernoulli coefficients, the LEO clock offsets at every of the \( K \) observed epochs and the GPS ambiguity parameters for \( m \) observed GPS satellites. The design matrix for the ionosphere-free carrier phase GPS-SST observations multiplied by the corrections to the approximations of the unknowns reads

\[ \left. \frac{\partial \Phi^s_{r,i}(\tau)}{\partial \mathbf{x}} \right|_{\mathbf{x} = \mathbf{x}_0} (\mathbf{x} - \mathbf{x}_0) = \begin{pmatrix} \frac{\partial \Phi^s_{r,i}(\tau)}{\partial \mathbf{x}_{eb}} & \frac{\partial \Phi^s_{r,i}(\tau)}{\partial \mathbf{x}_t} & \frac{\partial \Phi^s_{r,i}(\tau)}{\partial \mathbf{x}_A} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{eb} - \mathbf{x}_{eb,0} \\ \mathbf{x}_t - \mathbf{x}_{t,0} \\ \mathbf{x}_A - \mathbf{x}_{A,0} \end{pmatrix}. \quad (6.58) \]

The partial derivatives of the carrier phase observations with respect to the LEO clock offsets and the ambiguity parameters were discussed in Sec. 6.1.1.2. The partial derivatives of the carrier phase GPS-SST observations with respect to the Euler-Bernoulli coefficients can be performed by applying the chain rule.
Partial derivatives of the carrier phase observations with respect to the Euler-Bernoulli coefficients: The partial derivatives of ionosphere-free carrier phase SST observations between the GPS satellite $s$ and the GPS receiver $r$ on-board LEO with respect to the Euler-Bernoulli parameters can be written as follows,

$$
\frac{\partial \Phi^s r,3(\tau)}{\partial \mathbf{x}_{rb}} = \frac{\partial \Phi^s r,3(\tau)}{\partial \mathbf{r}(\tau)} \frac{\partial \mathbf{r}(\tau)}{\partial \mathbf{x}_{rb}},
$$

or in a matrix form,

$$
\mathbf{A}^s r,3(\tau) = \mathbf{a}^s r(\tau) \mathbf{A}^s r,3(\tau),
$$

given Eqs. (4.25) and (4.31),

$$
\mathbf{a}^s r(\tau) = \frac{\partial \Phi^s r,3(\tau)}{\partial \mathbf{v}(\tau)} = \left( e^s r, x(\tau) \; e^s r, y(\tau) \; e^s r, z(\tau) \right),
$$

$$
e^s r, x(\tau) = \frac{x_r(\tau) - x^s(\tau - \tau^s)}{\rho^s r(\tau)}, \quad e^s r, y(\tau) = \frac{y_r(\tau) - y^s(\tau - \tau^s)}{\rho^s r(\tau)}, \quad e^s r, z(\tau) = \frac{z_r(\tau) - z^s(\tau - \tau^s)}{\rho^s r(\tau)},
$$

and

$$
\mathbf{c}(\tau) := \begin{pmatrix} \mathbf{t}_r^T \mathbf{E}_c^{2J+1} & \mathbf{B}_c^{2J+1} \end{pmatrix} = \begin{pmatrix} c_1 & \cdots & c_{2J} \end{pmatrix},
$$

$$
\mathbf{t}_r = \begin{pmatrix} \tau \\ \vdots \\ \tau^{2J+1} \end{pmatrix}, \quad \mathbf{l} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Gauss-Markov model for SST carrier phase observations: The linearized Gauss-Markov model of the ionosphere-free carrier phase observations for the GPS satellite $s$ at epoch $\tau$ can be written as,

$$
\Delta \Phi^s r,3(\tau) = \mathbf{a}^s r(\tau) \mathbf{A}^s r,3(\tau) (\mathbf{x}_{rb} - \mathbf{x}_{rb}^0) + \mathbf{a}^s r(\tau) (\mathbf{x}_r - \mathbf{x}_r^0) + \mathbf{a}^s A(\tau) (\mathbf{x}_A - \mathbf{x}_{A,0}),
$$

with the weight of the ionosphere-free carrier phase observations from the LEO $r$ to the GPS satellite $s$,

$$
w^s r, w_3(\tau) = \frac{\sigma_0^2}{\sigma_{w_3 r}^2} \cos^2(z^r s(\tau)).
$$

If $m_r$ is the total number of carrier phase observations at time $\tau$ from the GPS receiver $r$ on-board LEO to GPS satellites $s_1, \ldots, s_{m_r}$ to $r$, the Gauss-Markov model for all unknowns reads,

$$
\begin{pmatrix}
\Delta \Phi^{s_1 r,3}(\tau) \\
\vdots \\
\Delta \Phi^{s_{m_r} r,3}(\tau)
\end{pmatrix} = \begin{pmatrix}
\mathbf{a}^{s_1 r}(\tau) \mathbf{A}^{s_1 r,3}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r} r}(\tau) \mathbf{A}^{s_{m_r} r,3}(\tau)
\end{pmatrix} (\mathbf{x}_{rb} - \mathbf{x}_{rb,0}) + \begin{pmatrix}
\mathbf{a}^{s_1 r}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r} r}(\tau)
\end{pmatrix} (\mathbf{x}_r - \mathbf{x}_{r,0}) + \begin{pmatrix}
\mathbf{a}^{s_1 A}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r} A}(\tau)
\end{pmatrix} (\mathbf{x}_A - \mathbf{x}_{A,0}),
$$

or

$$
\begin{pmatrix}
\Delta \Phi^{s_1 r,3}(\tau) \\
\vdots \\
\Delta \Phi^{s_{m_r} r,3}(\tau)
\end{pmatrix} = \begin{pmatrix}
\mathbf{a}^{s_1 r}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r} r}(\tau)
\end{pmatrix} \begin{pmatrix}
\mathbf{A}^{s_1 r,3}(\tau) \\
\vdots \\
\mathbf{A}^{s_{m_r} r,3}(\tau)
\end{pmatrix} (\mathbf{x}_{rb,0} - \mathbf{x}_{rb,0}) + \begin{pmatrix}
\mathbf{a}^{s_1 r}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r} r}(\tau)
\end{pmatrix} (\mathbf{x}_r - \mathbf{x}_{r,0}) + \begin{pmatrix}
\mathbf{a}^{s_1 A}(\tau) \\
\vdots \\
\mathbf{a}^{s_{m_r} A}(\tau)
\end{pmatrix} (\mathbf{x}_A - \mathbf{x}_{A,0}).
$$

(6.65)
or in matrix notation as,
\[
\Delta \mathbf{l}(\tau) = \mathbf{A}(\tau) \Delta \mathbf{x},
\]  
(6.66)

with the weight matrix of all ionosphere-free carrier phase observations at the epoch \(\tau\),
\[
\mathbf{W}(\tau) = \begin{pmatrix}
  w^{s_1}_{\tau,\Phi_3}(\tau) & \ldots & 0 & \ldots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \ldots & w^{s_2}_{\tau,\Phi_3}(\tau) & \ldots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \ldots & 0 & \ldots & w^{s_{m_\Phi}}_{\tau,\Phi_3}(\tau)
\end{pmatrix},
\]  
(6.67)

If the total number of the carrier phase observations at epochs \(\tau_1, \ldots, \tau_K\) is \(l(l \geq u)\), then the Gauss-Markov model for all tracked GPS satellites and all observed epochs reads,
\[
\begin{pmatrix}
  \Delta \mathbf{l}(\tau_1) \\
  \vdots \\
  \Delta \mathbf{l}(\tau_K)
\end{pmatrix} = \begin{pmatrix}
  \mathbf{A}(\tau_1) \\
  \vdots \\
  \mathbf{A}(\tau_K)
\end{pmatrix} \begin{pmatrix}
  \Delta \mathbf{x}
\end{pmatrix},
\]  
(6.68)

or
\[
\Delta \mathbf{l}(i) = \mathbf{A}(i \times u) \Delta \mathbf{x}(u), \quad \mathbf{W}_t = \begin{pmatrix}
  \mathbf{W}(\tau_1) & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & \mathbf{W}(\tau_K)
\end{pmatrix},
\]  
(6.69)

where it holds for the matrix dimensions
\(l\) the total number of carrier phase observations at epochs \(\tau_1, \ldots, \tau_K\),
\(u\) the total number of the unknowns as \(u = u_{eb} + u_t + u_A\),
\(u_{eb}\) the number of Euler-Bernoulli coefficients for an upper index \(J (u_{eb} = 6J)\),
\(u_t\) the total number of the LEO clock offsets (\(u_t = K - 1\)), except the first epoch,
\(u_A\) the GPS ambiguity parameters.

The corrections to the approximate unknown parameters and the \(a\)-posteriori variance-covariance matrix can be determined in the batch processing of all carrier phase SST observations according to
\[
\Delta \hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{W}_t \mathbf{A})^{-1} \mathbf{A}^T \mathbf{W}_t \Delta \mathbf{l}, \quad \mathbf{C}_{\Delta \hat{\mathbf{x}}} = (\mathbf{A}^T \mathbf{W}_t \mathbf{A})^{-1}.
\]  
(6.70)

The estimated unknowns in Eq. (6.70) are the corrections to the initial Euler-Bernoulli coefficients, the initial LEO clock offsets and the initial ambiguities of all observed GPS satellites from GPS receiver \(r\) on-board LEO. Because of the linearization, the estimation procedure has to be performed in an iterative manner. The convergence of the unknowns can be achieved after a few iterations as,
\[
\hat{\mathbf{x}}_{(i)} = \hat{\mathbf{x}}_{(i-1)} + \Delta \hat{\mathbf{x}}_{(i)} = \hat{\mathbf{x}}_{(i-1)} + \left(\mathbf{A}^T_{(i)} \mathbf{W}_t \mathbf{A}_{(i)}\right)^{-1} \mathbf{A}^T_{(i)} \mathbf{W}_t \Delta \mathbf{l}_{(i)}.
\]  
(6.71)

### 6.1.3 The Hybrid Case: Fourier Series and Euler-Bernoulli Polynomials

In the hybrid or combination version, the function \(d(\tau)\) is represented by a finite series in terms of Euler- and Bernoulli polynomials and a residual finite Fourier series,
\[
d(\tau) \equiv \bar{d}^0_{\phi}(\tau) + d^{\ell_{max}}_{\phi}(\tau).
\]  
(6.72)
Observed arc given by:
K discrete not necessarily equidistant positions, derived from a geometrical orbit determination strategy

\[ \mathbf{r}(\tau_k) = \mathbf{F}(\tau_k) + \mathbf{d}(\tau_k) \]

\( \tau_k \) with \( k = 1, 2, 3, \ldots, K \), \( \tau_1 \in [0, 1] \)

Determination of the Euler-Bernoulli coefficients up to degree \( j_{\text{max}} \) from the discrete satellite’s positions by a least squares adjustment - Eqs. (6.50) and (6.53) without correcting the boundary vectors

\[ \mathbf{l} = \mathbf{A} \mathbf{x}, \]

\[ \mathbf{l} = \begin{bmatrix} \mathbf{d}(\tau_1) \\ \vdots \\ \mathbf{d}(\tau_K) \end{bmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{T}^T \left( \mathbf{E}_{1,\text{max}}^T \right)^T \\ \mathbf{b}_{2,\text{max}}^T \end{pmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{e}_{2,\text{max}} \\ \mathbf{b}_{2,\text{max}} \end{bmatrix} \]

\[ \hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{C} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}, \quad \mathbf{C}_s = (\mathbf{A}^T \mathbf{C} \mathbf{A})^{-1}. \]

\[ \mathbf{l} = \mathbf{A} \mathbf{x}, \]

\[ \mathbf{l} = \begin{bmatrix} \mathbf{r}(\tau_1) \\ \vdots \\ \mathbf{r}(\tau_K) \end{bmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b}^T \end{pmatrix}, \quad \mathbf{x} = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_s \end{bmatrix} \]

\[ \hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{C} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{d}, \quad \mathbf{C}_s = (\mathbf{A}^T \mathbf{C} \mathbf{A})^{-1}. \]

Determination of the Fourier coefficients up to degree \( n \) from the discrete satellites’s positions by a least squares adjustment - Eqs. (6.18) and (6.21)

\[ \mathbf{d}(\tau) = \mathbf{d} - \mathbf{T} (\mathbf{E}_{1,\text{max}}^T \mathbf{T}^T \mathbf{b}_{2,\text{max}}^T) \begin{bmatrix} \mathbf{e}_{2,\text{max}} \\ \mathbf{b}_{2,\text{max}} \end{bmatrix} \]

\[ \mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_s \end{bmatrix}, \quad \mathbf{d}(\tau) = \begin{bmatrix} \mathbf{d}_1 \\ \vdots \\ \mathbf{d}_s \end{bmatrix}. \]

Solution

\[ \mathbf{r}(\tau) = \mathbf{F}(\tau) + \mathbf{d}(\tau) + \mathbf{d}(\tau) \]

\[ \mathbf{F}(\tau) = \sin \mu (\mathbf{l} - \mathbf{r}) + \sin \mu \mathbf{r} - \mathbf{r} \]

\[ \mathbf{d}(\tau) = \sum_{j=1}^{l_{\text{max}}} \mathbf{e}_j (\tau) + \sum_{j=1}^{l_{\text{max}}} \mathbf{b}_{2,j} (\tau) \]

\[ \mathbf{d}(\tau) = \sum_{i=1}^{v_{\text{max}}} \bar{\mathbf{d}}_i \sin(v \pi \tau) \]

Figure 6.1: Computation scheme of the kinematical orbit determination procedure, with Euler-Bernoulli coefficients calculated from given LEO absolute positions.
6.1. Kinematical Orbit Determination

Observed arc given by:
K discrete not necessarily equidistant positions, derived from a geometrical orbit determination strategy

\[ r(\tau_k) = \bar{r}(\tau_k) + d(\tau_k) \]

\[ \tau_k \text{ with } k=1,2,3,\ldots,K. \quad \tau_k \in [0,1] \]

Figure 6.2: Computation scheme of the kinematical orbit determination procedure, with Euler-Bernoulli coefficients calculated from amplitudes of Fourier series.
The residual Fourier series is restricted by an upper index \( \bar{n} \) and the Euler-Bernoulli polynomials by an upper index \( J_{\text{max}} \). As demonstrated in Sec. 5.4.1.3, the approximation quality of a combination of both series is superior compared to the approximation of either one of these series as discussed in the Secs. 6.1.1 and 6.1.2.

Because of the fact that both orbit representations could be used exclusively, the combination of both superior compared to the approximation of either one of these series as discussed in the Secs. 6.1.1 and 6.1.2.

\[ \text{With the Euler-Bernoulli coefficients, the Euler-Bernoulli part of the difference function can be derived based on the sine analysis of the difference function,} \]

\[ \text{The smoothness is defined by the degree of differentiability of the periodically continued function} \text{. If the (unknown) real orbit and the Euler-Bernoulli polynomials have identical derivatives up to a high degree, then the difference function} \text{ shows differentiability up to this degree as well. The convergence behaviour} \]

\[ \text{The upper degree} \bar{\tau} \text{for the epochs conditions at the boundaries of this function, if it is periodically continued. The system of equations reads} \]

\[ \begin{bmatrix} \bar{d}(\tau_1) \\ \vdots \\ \bar{d}(\tau_K) \end{bmatrix} = \mathbf{d} = \mathbf{S}^T \mathbf{d}_n - \pi^T \left( \begin{bmatrix} \mathbf{E}^{2J_{\text{max}}+1}_n \end{bmatrix}^T \begin{bmatrix} \mathbf{B}^{2J_{\text{max}}+1}_n \end{bmatrix}^T \right) \begin{bmatrix} \mathbf{e}^{2J_{\text{max}}}_n \\ \mathbf{b}^{2J_{\text{max}}+1}_n \end{bmatrix}, \]

\[ \text{and the Fourier amplitudes up to the index} \bar{n} \text{ are derived based on the sine analysis of the difference function,} \]

\[ \begin{bmatrix} \bar{d}_1 \\ \vdots \\ \bar{d}_{\bar{n}} \end{bmatrix} = (\mathbf{S}^T \mathbf{S})^{-1} \begin{bmatrix} \bar{d}(\tau_1) \\ \vdots \\ \bar{d}(\tau_K) \end{bmatrix}, \quad \mathbf{S}^T = \begin{bmatrix} \sin(\pi \tau_1) & \cdots & \sin(\bar{n} \pi \tau_1) \\ \vdots & \ddots & \vdots \\ \sin(\pi \tau_K) & \cdots & \sin(\bar{n} \pi \tau_K) \end{bmatrix}. \]
of the Fourier series \( \tilde{d}_n^\ell \mathbf{r} (\tau) \) is closely related to this degree of differentiability. Because of the fact that the Euler-Bernoulli polynomials are constructed such that the derivatives of the satellite’s arcs coincide with those of the Euler-Bernoulli polynomials, the coincidence should be realized up to a high degree. But this holds only theoretically, because the quality of the approximation depends on the maximum index \( J_{\text{max}} \) of \( d_{\text{max}}^\ell \mathbf{r} (\tau) \) and of the accuracy of the real orbit which is known only point-wise by carrier phase GPS-SST observations. The Fig. 5.13 shows the residual functions \( \tilde{d}(\tau) = d(\tau) - d_{\text{max}}^\ell \mathbf{r} (\tau) \) for various upper indices \( J_{\text{max}} = 1, 2, 3, 4, 5 \) and Fig. 5.19 shows the residuals of the sum of both functions for the Euler-Bernoulli up to \( J_{\text{max}} = 4 \) and different upper indices \( n \) compared to the error-free values in the ellipse mode.

### 6.2 Reduced-Kinematical and Dynamical Orbit Determination

The kinematical orbit determination is based on discrete measurements, either positions derived from a GNSS processing strategy as described in chapter 4, or used directly as observations to derive the parameters of any of the kinematical orbit determination modes described in Sec. 6.1. The kinematic orbit parameters contain no dynamic information of the force function model; they are only based on empirical observations, taken at discrete epochs of the satellite’s motion along the orbit. In the following, we will extend the hybrid case of the kinematical orbit determination procedures as treated in Sec. 6.1.3 in such a way that dynamical restrictions can be introduced in the orbit determination procedure. This dynamical information is contained in the orbit coefficients \( \mathbf{d} \), which are related to the force function according to Eqs. (5.13), (5.23) or (5.34) and the respective reference motion types Eqs. (5.10), (5.20) or (5.37). To show the principle behind this idea, we demonstrate it in case of Eq. (5.23),

\[
\tilde{d}_n = -\frac{2T^2}{\nu^2\pi^2} \int_0^1 \sin(\nu \pi \tau) a^I(\tau; \mathbf{r}, \dot{\mathbf{r}}) d\tau.
\]

If we restrict the dynamical model to the gravitational field of the Earth, expressed by a spherical harmonics expansion up to degree \( l_{\text{max}} \), then it holds,

\[
a^I(\tau; \mathbf{r}, \dot{\mathbf{r}}) = a(\tau; \mathbf{r}, \dot{\mathbf{r}}) = \frac{GM}{\alpha^2} \mathbf{r}(\tau), \quad a(\tau; \mathbf{r}, \dot{\mathbf{r}}) = \nabla V(\tau; \mathbf{r}, \dot{\mathbf{r}}),
\]

with the gravitational potential,

\[
V = \frac{GM}{\| \mathbf{r} \|} \sum_{l=0}^{l_{\text{max}}} \sum_{m=0}^{l} \left( \frac{R}{\| \mathbf{r} \|} \right)^l (c_{l m} C_{l m}(\vartheta, \lambda) + s_{l m} S_{l m}(\vartheta, \lambda))
\]

the surface spherical harmonics of degree \( l \) and order \( m \),

\[
C_{l m}(\vartheta, \lambda) = P_l^m(\cos \vartheta) \sin(m \lambda), \quad S_{l m}(\vartheta, \lambda) = P_l^m(\cos \vartheta) \cos(m \lambda)
\]

and the associated Legendre functions \( P_l^m(\cos \vartheta) \). The dynamical restrictions can be introduced in the system of observation equations as a-priori information and an appropriate a-priori variance-covariance matrix. If we consider for simplification the case of pseudo observed positions, then it reads according to Eq. (6.18),

\[
\begin{bmatrix}
\mathbf{r}(\tau_1) \\
\vdots \\
\mathbf{r}(\tau_K) \\
\mathbf{d}_i \\
\vdots \\
\mathbf{d}_j \\
\mathbf{d}_n
\end{bmatrix}
\begin{bmatrix}
\mathbf{a}(\tau_1) \\
\vdots \\
\mathbf{a}(\tau_K) \\
\mathbf{b}(\tau_1) \\
\vdots \\
\mathbf{b}(\tau_K) \\
\sin(\pi \tau_1) \\
\vdots \\
\sin(\pi \tau_K) \\
\sin(j \pi \tau_1) \\
\vdots \\
\sin(j \pi \tau_K) \\
\sin(n \pi \tau_1) \\
\vdots \\
\sin(n \pi \tau_K)
\end{bmatrix}
\begin{bmatrix}
r(\tau_1) \\
\vdots \\
r(\tau_K) \\
d_i \\
\vdots \\
d_j \\
d_n
\end{bmatrix}
\]
with the variance-covariance matrix

$$\mathbf{C}_l = \begin{pmatrix}
   \mathbf{C}(\tau_1) & \cdots & 0 & 0 & \cdots & 0 \\
   \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
   0 & \cdots & \mathbf{C}(\tau_K) & 0 & \cdots & 0 \\
   \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
   0 & \cdots & 0 & \mathbf{C}(\tilde{d}_i) & \cdots & 0 \\
   \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
   0 & \cdots & 0 & 0 & \cdots & \mathbf{C}(\tilde{d}_j)
\end{pmatrix}$$  \tag{6.81}

The quantities \(\tilde{d}_i\) to \(\tilde{d}_j\) are considered as a-priori information with the variance-covariance matrices \(\mathbf{C}(\tilde{d}_i)\) to \(\mathbf{C}(\tilde{d}_j)\). The indices \(i\) and \(j\) define the sine coefficients with a-priori information and can be selected so that they cover the total index domain \(i = 1\) and \(j = n\) or some specific amplitudes. If we split up the design matrix Eq. (6.80) with the variance-covariance matrix \(\mathbf{C}_l\),

$$\mathbf{l} = \mathbf{A}\mathbf{x},$$  \tag{6.82}

in the following way (do not mix up the observation column matrix \(\mathbf{l}\) in Eq. (6.82) with the unit matrix \(\mathbf{l}\) in Eq. (6.83)),

$$\begin{pmatrix}
   \mathbf{l}_1 \\
   \mathbf{l}_2
\end{pmatrix} = \begin{pmatrix}
   \mathbf{A}_1 & \mathbf{A}_2 \\
   \mathbf{0} & \mathbf{l}
\end{pmatrix} \begin{pmatrix}
   \mathbf{x}_1 \\
   \mathbf{x}_2
\end{pmatrix},$$  \tag{6.83}

with the a-priori variance-covariance matrix Eq. (6.81),

$$\mathbf{C}_l = \begin{pmatrix}
   \mathbf{C}_1 & \mathbf{0} \\
   \mathbf{0} & \mathbf{C}_2
\end{pmatrix},$$  \tag{6.84}

with

$$\begin{pmatrix}
   \mathbf{l}_1 \\
   \mathbf{l}_2
\end{pmatrix} = \begin{pmatrix}
   \mathbf{r}(\tau_1) & \cdots & \mathbf{r}(\tau_K)
\end{pmatrix}^T, \quad \mathbf{l}_2 = \begin{pmatrix}
   \tilde{d}_i \\
   \cdots \\
   \tilde{d}_j
\end{pmatrix}^T, \quad \mathbf{A}_1 = \begin{pmatrix}
   \mathbf{a} & \mathbf{b}
\end{pmatrix}, \quad \mathbf{A}_2 = \mathbf{S}^T,$$  \tag{6.85}

$$\mathbf{C}_1 = \begin{pmatrix}
   \mathbf{C}(\tau_1) & \cdots & 0 \\
   \vdots & \ddots & \vdots \\
   0 & \cdots & \mathbf{C}(\tau_K)
\end{pmatrix}, \quad \mathbf{C}_2 = \mathbf{C}(\tilde{d}) = \begin{pmatrix}
   \mathbf{C}(\tilde{d}_i) & \cdots & 0 \\
   \vdots & \ddots & \vdots \\
   0 & \cdots & \mathbf{C}(\tilde{d}_j)
\end{pmatrix}.$$  \tag{6.86}

The least squares solution reads (ILK 1977, Eq. (90))

$$\begin{pmatrix}
   \mathbf{x}_1 \\
   \mathbf{x}_2
\end{pmatrix} = \mathbf{N}^{-1} \begin{pmatrix}
   \mathbf{A}_1^T \mathbf{C}_1^{-1} \mathbf{l}_1 \\
   \mathbf{A}_2^T \mathbf{C}_1^{-1} \mathbf{l}_1 + \mathbf{C}_2^{-1} \mathbf{l}_2
\end{pmatrix},$$  \tag{6.87}

with the inverse normal matrix

$$\mathbf{N}^{-1} = \begin{pmatrix}
   \mathbf{Q}_{x_1 x_1} & \mathbf{Q}_{x_1 x_2} & \mathbf{Q}_{x_2 x_1} & \mathbf{Q}_{x_2 x_2}
\end{pmatrix},$$  \tag{6.88}

and its sub-matrices

$$\mathbf{Q}_{x_1 x_1} = (\mathbf{A}_1^T \mathbf{C}_1^{-1} \mathbf{A}_1)^{-1},$$  \tag{6.89}

$$\mathbf{Q}_{x_1 x_2} = -\mathbf{Q}_{x_1 x_1} \mathbf{A}_1^T \mathbf{C}_1^{-1} \mathbf{A}_2 \mathbf{C}_2,$$  \tag{6.90}

$$\mathbf{Q}_{x_2 x_1} = -\mathbf{C}_2 \mathbf{A}_2^T \mathbf{C}_1^{-1} \mathbf{A}_1 \mathbf{Q}_{x_1 x_1} \mathbf{A}_1^T \mathbf{C}_1^{-1} \mathbf{A}_2 \mathbf{C}_2,$$  \tag{6.91}

$$\mathbf{Q}_{x_2 x_2} = \mathbf{C}_2 - \mathbf{C}_2 \mathbf{A}_2^T \mathbf{C}_1^{-1} \mathbf{A}_2 \mathbf{C}_2 + \mathbf{C}_2 \mathbf{A}_2^T \mathbf{C}_1^{-1} \mathbf{A}_1 \mathbf{Q}_{x_1 x_1} \mathbf{A}_1^T \mathbf{C}_1^{-1} \mathbf{A}_2 \mathbf{C}_2,$$  \tag{6.92}

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{A}_2 \mathbf{C}_2 \mathbf{A}_2^T.$$  \tag{6.93}