The integral kernel \( K^{II}(\tau, \tau') \) can be expressed by its eigen values and its eigen functions,
\[
K^{II}(\tau, \tau') = 2\sum_{\nu=1}^{\infty} \sin(\nu\pi\tau) \sin(\nu\pi\tau') \frac{2T^2}{\nu^2\pi^2 - \mu^2},
\]
and the coefficients can be derived by inserting Eq. (5.19) and Eq. (5.21) in Eq. (5.16) and comparison of the coefficients with the same eigen functions,
\[
\sum_{\nu=1}^{\infty} c_{\nu} \sin(\nu\pi\tau) = -\sum_{\nu=1}^{\infty} \sin(\nu\pi\tau) \frac{2T^2}{\nu^2\pi^2 - \mu^2} \int_{\tau'=0}^{1} \sin(\nu\pi\tau') a^{II}(\tau; r, \dot{r}) d\tau'.
\]
Then the coefficients read,
\[
c_{\nu} = -\frac{2T^2}{\nu^2\pi^2 - \mu^2} \int_{\tau'=0}^{1} \sin(\nu\pi\tau') a^{II}(\tau; r, \dot{r}) d\tau'.
\]

5.2.3 Solution of the Equation of Motion as Correction to a Reference Orbit

If the force function \( \tilde{a}(\tau'; \tilde{r}, \dot{\tilde{r}}) := a(\tau'; r, \dot{r}) \) in Eq. (5.1) is well-known, then the satellite arc can be determined in the dynamic orbit determination mode. In this case the orbit can be represented in a semi-analytical way by Eq. (5.9)
\[
\tilde{r}(\tau) = \bar{\tilde{r}}(\tau) + \sum_{\nu=1}^{\infty} \tilde{c}_{\nu} \sin(\nu\pi\tau) = \bar{\tilde{r}}(\tau) + \tilde{d}(\tau)
\]
with (Eq. (5.10))
\[
\bar{\tilde{r}}(\tau) = (1 - \tau) \tilde{r}_A + \tau \tilde{r}_B,
\]
and the coefficients (Eq. (5.13))
\[
\tilde{c}_{\nu} = -\frac{2T^2}{\nu^2\pi^2 - \mu^2} \int_{\tau'=0}^{1} \sin(\nu\pi\tau') \tilde{a}(\tau'; \tilde{r}, \dot{\tilde{r}}) d\tau'.
\]
Instead of Eq. (5.10) also Eq. (5.20) in case of the linear extended Newton operator could be selected alternatively,
\[
\tilde{r}(\tau) = \frac{\sin\mu(1 - \tau)}{\sin\mu} \tilde{r}_A + \frac{\sin\mu\tau}{\sin\mu} \tilde{r}_B.
\]
Based on the dynamical reference orbit \( \bar{\tilde{r}}(\tau) \), following the force function model \( \tilde{a}(\tau'; \tilde{r}, \dot{\tilde{r}}) \), the dynamical orbit for an improved force function model
\[
a(\tau'; r, \dot{r}) = \tilde{a}(\tau'; \tilde{r}, \dot{\tilde{r}}) + \Delta a(\tau'; \tilde{r} + \tilde{x}, \dot{\tilde{r}} + \dot{\tilde{x}}),
\]
can be formulated as difference motion,
\[
r(\tau) = x(\tau) + \tilde{r}(\tau).
\]
Based on the simple Newton operator \( L \),
\[
L := L_N = \frac{d}{dt} \dot{x}(t),
\]
the equation of motion for the difference motion \( \mathbf{x}(\tau) \) reads (Fig. 5.3),

\[
\frac{d}{dt} \mathbf{\dot{x}}(t) = \mathbf{a}(\tau'; \mathbf{r}, \mathbf{\dot{r}}) - \mathbf{\tilde{a}}(\tau'; \mathbf{\tilde{r}}, \mathbf{\dot{\tilde{r}}}) = \Delta \mathbf{a}(\tau'; \mathbf{\tilde{r}} + \mathbf{x}, \mathbf{\dot{\tilde{r}}} + \mathbf{\dot{x}}). \tag{5.31}
\]

The solution of the boundary value problem with the corrections \( \mathbf{\bar{x}}_A, \mathbf{\bar{x}}_B \) to the boundary values \( \mathbf{r}_A, \mathbf{r}_B \) reads

\[
\mathbf{x}(\tau) = \mathbf{\bar{x}}(\tau) + \mathbf{d}(\tau) = \mathbf{\bar{x}}(\tau) + \sum_{\nu=1}^{\infty} \mathbf{x}_\nu \sin(\nu \pi \tau), \tag{5.32}
\]

with

\[
\mathbf{\bar{x}}(\tau) = (1 - \tau) \mathbf{x}_A + \tau \mathbf{x}_B, \tag{5.33}
\]

and the corrections to the coefficients \( c_\nu \) of Eq. (5.13) or Eq. (5.23),

\[
\mathbf{x}_\nu = -\frac{2T^2}{\nu^2 \pi^2} \int_{\tau' = 0}^{1} \sin(\nu \pi \tau') \Delta \mathbf{a}(\tau'; \mathbf{\tilde{r}} + \mathbf{x}, \mathbf{\dot{\tilde{r}}} + \mathbf{\dot{x}}) \, d\tau', \tag{5.34}
\]

derived analogously to Eq. (5.13) or Eq. (5.23). The final solution can be written as follows

\[
\mathbf{r}(\tau) = (\mathbf{\bar{r}}(\tau) + \mathbf{\bar{x}}(\tau)) + \sum_{\nu=1}^{\infty} (\mathbf{\tilde{c}}_\nu + \mathbf{x}_\nu) \sin(\nu \pi \tau), \tag{5.35}
\]

corresponding to Eq. (5.9) or Eq. (5.19),

\[
\mathbf{r}(\tau) = \mathbf{\bar{r}}(\tau) + \sum_{\nu=1}^{\infty} \mathbf{x}_\nu \sin(\nu \pi \tau) = \mathbf{\bar{r}}(\tau) + \mathbf{d}(\tau), \tag{5.36}
\]

with

\[
\mathbf{\bar{r}}(\tau) := \mathbf{\bar{r}}(\tau) + \mathbf{\bar{x}}(\tau) = \mathbf{\bar{r}}(\tau) + \mathbf{\bar{x}}(\tau) + \mathbf{\tilde{d}}(\tau), \tag{5.37}
\]

and

\[
\mathbf{c}_\nu := \mathbf{\tilde{c}}_\nu + \mathbf{x}_\nu. \tag{5.38}
\]
5.2.4 Interpretation of the Solution of Fredholm’s Integral Equation as Fourier Series

It can be shown (Ilk 1976), that the solution series Eq. (5.9) or Eq. (5.19) contain Fourier series of the difference function,

\[ d(\tau) := r(\tau) - \bar{r}(\tau) = \sum_{\nu=1}^{\infty} d_{\nu} \sin(\nu \pi \tau) =: d_{\infty} F(\tau), \quad (5.39) \]

with the coefficients \( d_{\nu} \) derived from the difference function,

\[ d_{\nu} = 2 \int_{\tau'=0}^{1} d(\tau') \sin(\nu \pi \tau') \, d\tau'. \quad (5.40) \]

It should be pointed out that the reference motion \( \bar{r}(\tau) \) can be also of the type of Eq. (5.37),

\[ \bar{r}(\tau) = \tilde{r}(\tau) + \bar{x}(\tau). \quad (5.41) \]

Because of the uniqueness of both series in Eqs.(5.9), (5.19) and (5.39), it holds

\[ d_{\nu} = c_{\nu} \text{ for } \nu = 1, \ldots, \infty. \quad (5.42) \]

If the function \( d(\tau) \) is continued to an odd periodic function with the period \( 2T \) or normalized to the interval \([-1,1]\), then it holds

\[ d(-\tau) = -d(\tau), \quad (5.43) \]

and all cosine terms become zero. Some important facts result from the theory of Fourier series and hold also for the solution of the boundary value problem as well:

- Because of the fact that the function \( d(\tau) \) is continuous in \([-1,1]\) together with its first derivatives, the series \( D_{\infty}(\tau) \) converges absolutely and uniformly and therefore represents the function \( d(\tau) \); the coefficients \( d_{\nu,i} \) for \( i = 1, 2, 3 \) build a zero sequence:
  \[ \lim_{\nu \to \infty} d_{\nu,i} = 0, \quad (5.44) \]

- Because of the fact that the functions \( d_i(\tau) \) are continuous within \([-1,1]\), the Fourier coefficients fulfill Parseval’s equation,
  \[ \sum_{\nu=1}^{\infty} d_{\nu,i}^2 = 2 \int_{0}^{1} d_i^2(\tau) \, d\tau, \quad (5.45) \]

- The asymptotic behavior of the coefficients depends on the order of differentiability of the function \( d(\tau) \). Because of the specific characteristics of this function, it holds:
  \[ |d_{\nu,i}| \leq \frac{c}{\nu^3}, \quad (5.46) \]
  with a constant \( c > 0 \).

- The limitation of the series \( D_{\infty}(\tau) \) at a finite index \( n \),
  \[ D_n(\tau) = \sum_{\nu=1}^{n} d_{\nu} \sin(\nu \pi \tau), \quad (5.47) \]
is the best mean approximation of the function $d(\tau)$ in the interval $\tau \in [-1, 1]$.

The minimal quadratic error for the coordinates reads

$$
\frac{1}{2} \int_0^1 (d_i(\tau) - D_{n,i}(\tau))^2 d\tau = \frac{1}{2} \int_0^1 (d_i(\tau))^2 d\tau - \frac{1}{2} \sum_{\nu=1}^n d_{\nu,i}^2,
$$

and it holds for the residual terms of the series

$$
\lim_{n \to \infty} \frac{1}{2} \int_0^1 (d_i(\tau) - D_{n,i}(\tau)) d\tau = 0.
$$

The Fourier coefficients $d_\nu$ of the function $d(\tau)$,

$$
d_\nu = 2 \int_{\tau'=0}^{1} d(\tau) \sin(\nu \pi \tau') d\tau',
$$

can be determined approximately by

$$
d_\nu \approx \frac{2}{K+1} \sum_{k=1}^K d(\tau_k) \sin\left(\frac{\nu \pi k}{K+1}\right),
$$

if the function $d(\tau)$ is given discrete at a regular sampling rate of $\tau_i$,

$$
d(\tau_k) \quad \text{with} \quad \tau_k = \frac{k}{K+1}, \quad k = 1, 2, 3, \ldots, K, \quad \tau_k \in ]0, 1[.
$$

This equidistant ephemeris of satellite positions $d(\tau)$ can be considered to be given, derived by a geometrical orbit determination procedure based on carrier phase GPS-SST measurements.

$$
d(\tau_k) := r(\tau_k) - \bar{r}(\tau_k) \rightarrow \begin{pmatrix} d_{k,1} \\ d_{k,2} \\ d_{k,3} \end{pmatrix} = \begin{pmatrix} x_r(\tau_k) + \bar{x}_r(\tau_k) \\ y_r(\tau_k) + \bar{y}_r(\tau_k) \\ z_r(\tau_k) + \bar{z}_r(\tau_k) \end{pmatrix}.
$$

The arc is divided by these points in $K+1$ equal parts of the arc. If the force function is known then the coefficients fulfill the Eq. (5.23) with the reduced force Eq. (5.18).

### 5.3 Series of Euler–Bernoulli Polynomials

#### 5.3.1 Continuous Position Function

##### 5.3.1.1 From Fourier Series to Series of Euler and Bernoulli Polynomials

The orbit of a satellite can be represented in a semi-analytical way by one of the Eqs. (5.9), (5.19) or (5.36),

$$
r(\tau) = \bar{r}(\tau) + \sum_{\nu=1}^\infty c_\nu \sin(\nu \pi \tau).
$$

An alternative representation of the orbit can be derived starting from the Eq. (5.39) with the Fourier coefficients of Eq. (5.40). As already outlined, Eq. (5.39) can be written as

$$
d(\tau) := r(\tau) - \bar{r}(\tau) = \sum_{\nu=1}^{\infty} d_\nu \sin(\nu \pi \tau) = \sum_{\nu=1}^{\infty} c_\nu \sin(\nu \pi \tau),
$$
so that it holds
\[ c_\nu \triangleq d_\nu = 2 \int_{\tau'=-1}^{1} d(\tau') \sin(\nu \pi \tau') \, d\tau'. \] (5.56)

If we apply a first integration by parts to this equation then we can write:
\[ d_\nu = 2 \int_{\tau'=-1}^{1} d(\tau') \sin(\nu \pi \tau') \, d\tau' = -2d(\tau') \frac{1}{\nu \pi} \cos(\nu \pi \tau') \bigg|_{\tau'=-1}^{1} + 2 \frac{1}{\nu \pi} \int_{\tau'=-1}^{1} d^{[1]}(\tau') \cos(\nu \pi \tau') \, d\tau', \] (5.57)
and by inserting the limits and considering \( d(\tau' = 0) = d(\tau' = 1) = 0, \)
\[ d_\nu = 2 \frac{1}{\nu \pi} \int_{\tau'=-1}^{1} d^{[1]}(\tau') \cos(\nu \pi \tau') \, d\tau' = \frac{2}{\nu \pi} \int_{\tau'=-1}^{1} d^{[2]}(\tau') \sin(\nu \pi \tau') \, d\tau'. \] (5.58)
with the first derivative of the orbit \( d^{[1]}(\tau) \) with respect to the normalized time \( \tau \). A second integration by parts results in
\[ d_\nu = 2 \frac{1}{\nu \pi} \int_{\tau'=-1}^{1} d^{[1]}(\tau') \cos(\nu \pi \tau') \, d\tau' = -2 \frac{1}{(\nu \pi)^2} \int_{\tau'=-1}^{1} d^{[2]}(\tau') \sin(\nu \pi \tau') \, d\tau'. \] (5.59)
A third integration by parts gives
\[ d_\nu = -2 \frac{1}{(\nu \pi)^2} \int_{\tau'=-1}^{1} d^{[2]}(\tau') \sin(\nu \pi \tau') \, d\tau' = 2 d^{[2]}(\tau') \frac{1}{(\nu \pi)^3} \cos(\nu \pi \tau') \bigg|_{\tau'=-1}^{1} + 2 \frac{1}{(\nu \pi)^3} \int_{\tau'=-1}^{1} d^{[3]}(\tau') \cos(\nu \pi \tau') \, d\tau', \] (5.60)
and a fourth integration by parts,
\[ d_\nu = 2 d^{[2]}(\tau') \frac{1}{(\nu \pi)^3} \cos(\nu \pi \tau') \bigg|_{\tau'=-1}^{1} + 2 \frac{1}{(\nu \pi)^4} \int_{\tau'=-1}^{1} d^{[4]}(\tau') \sin(\nu \pi \tau') \, d\tau'. \] (5.61)
The general expression reads after \( 2J + 2 \) integrations by parts (Klose 1985),
\[ d_\nu = \sum_{j=1}^{J} d^{[2j]}(\tau') \frac{2(-1)^{j+1}}{(\nu \pi)^{2j+1}} \cos(\nu \pi \tau') \bigg|_{\tau'=-1}^{1} + \beta \frac{2}{(\nu \pi)^{2J+2}} \int_{\tau'=-1}^{1} d^{[2J+2]}(\tau') \sin(\nu \pi \tau') \, d\tau', \] (5.62)
with
\[ \beta = \begin{cases} +1 & J = 1, 4, 5, 8, 9, 12, 13, \\ -1 & J = 2, 3, 6, 7, 10, 11, \\ \\ \end{cases}, \] (5.63)
and with the integration limits in the first term,
\[ d_\nu = \sum_{j=1}^{J} \frac{2(-1)^{j+1}}{(\nu \pi)^{2j+1}} \left( (-1)^{j} d^{[2j]}(1) - d^{[2j]}(0) \right) + \beta \frac{2}{(\nu \pi)^{2J+2}} \int_{\tau'=-1}^{1} d^{[2J+2]}(\tau') \sin(\nu \pi \tau') \, d\tau'. \] (5.64)
Inserting Eq. (5.64) in Eq. (5.39) gives

\[ \mathbf{d}(\tau) := \mathbf{r}(\tau) - \mathbf{F}(\tau) = \sum_{\nu=1}^{\infty} d_{\nu} \sin(\nu \pi \tau) = \]

\[ = \sum_{\nu=1}^{\infty} \left( \sum_{j=1}^{J} \frac{2(-1)^{j+1}}{(\nu \pi)^{2j+1}} \left( (-1)^{\nu} d^{[2j]}(1) - d^{[2j]}(0) \right) + \beta \frac{2}{(\nu \pi)^{2j+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(\nu \pi \tau') \, d\tau' \right) \sin(\nu \pi \tau). \]  

(5.65)

The sine coefficients read in this case as follows:

\[ d_{\nu} = \sum_{j=1}^{J} \frac{2(-1)^{j+1}}{(\nu \pi)^{2j+1}} \left( (-1)^{\nu} d^{[2j]}(1) - d^{[2j]}(0) \right) + \beta \frac{2}{(\nu \pi)^{2j+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(\nu \pi \tau') \, d\tau'. \]  

(5.66)

If we separate the inner sum of Eq. (5.65) in terms of even and odd indices \( \nu \),

\[ \mathbf{d}(\tau) := \mathbf{r}(\tau) - \mathbf{F}(\tau) = \sum_{\nu=1}^{\infty} d_{\nu} \sin(\nu \pi \tau) = \]

\[ = \sum_{\nu=1}^{J} \frac{2(-1)^{j+1}}{(2\nu \pi)^{2j+1}} \left( d^{[2j]}(1) - d^{[2j]}(0) \right) \sum_{\nu=1}^{\infty} \frac{\sin(2\nu \pi \tau)}{\nu^{2j+1}} + \]

\[ + \sum_{\nu=1}^{J} \frac{2(-1)^{j}}{(\nu \pi)^{2j+1}} \left( d^{[2j]}(1) + d^{[2j]}(0) \right) \sum_{\nu=1}^{\infty} \frac{\sin(2\nu \pi - 1) \pi \tau}{(2\nu - 1)^{2j+1}} + \]

\[ + \beta \frac{2}{(2\nu \pi)^{2j+2}} \sum_{\nu=1}^{\infty} \left( \frac{1}{\nu^{2j+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(\nu \pi \tau') \, d\tau' \sin(\nu \pi \tau) \right), \]

so that it holds for the coefficients according to Eq. (5.66) here for even coefficients,

\[ d_{2\nu} = \sum_{j=1}^{J} \frac{2(-1)^{j+1}}{(2\nu \pi)^{2j+1}} \left( d^{[2j]}(1) - d^{[2j]}(0) \right) + \beta \frac{2}{(2\nu \pi)^{2j+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(2\nu \pi \tau') \, d\tau', \]

(5.67)

and for odd coefficients

\[ d_{2\nu-1} = \sum_{j=1}^{J} \frac{2(-1)^{j}}{(2\nu - 1)^{2j+1}\pi^{2j+1}} \left( d^{[2j]}(1) + d^{[2j]}(0) \right) + \beta \frac{2}{(2\nu - 1)^{2j+2}\pi^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(2\nu - 1) \pi \tau' \, d\tau'. \]

(5.68)

The terms in Eq. (5.67) can be replaced by the absolutely and uniformly continuous series expansions of the Euler polynomials (Abramowitz and Stegun 1972),

\[ E_{2j}(\tau) = \frac{4(-1)^{j}(2j)!}{\pi^{2j+1}} \sum_{\nu=1}^{\infty} \frac{\sin(\nu \pi - 1) \pi \tau}{(2\nu - 1)^{2j+1}}, \]

(5.69)

and the Bernoulli polynomials

\[ B_{2j+1}(\tau) = \frac{2(-1)^{j+1}(2j + 1)!}{(2\pi)^{2j+1}} \sum_{\nu=1}^{\infty} \frac{\sin(\nu \pi \tau)}{\nu^{2j+1}}. \]

(5.70)
The Euler polynomials of even degree can be represented also as polynomials of the normalized time $\tau$ as follows, e.g. for $j = 1, 2, 3, 4, 5, 6$ (Abramowitz and Stegun 1972):

\[
E_2(\tau) = -\tau + \tau^2,
\]
\[
E_4(\tau) = -2\tau^3 + \tau^4,
\]
\[
E_6(\tau) = -3\tau + 5\tau^3 - 3\tau^5 + \tau^6,
\]
\[
E_8(\tau) = 17\tau - 28\tau^3 + 14\tau^5 - 4\tau^7 + \tau^8,
\]
\[
E_{10}(\tau) = -155\tau + 255\tau^3 - 126\tau^5 + 30\tau^7 - 5\tau^9 + \tau^{10},
\]
\[
E_{12}(\tau) = 2073\tau - 3410\tau^3 + 1683\tau^5 - 396\tau^7 + 55\tau^9 - 6\tau^{11} + \tau^{12},
\]

and the Bernoulli polynomials for odd degrees read for the indices $j = 1, 2, 3, 4, 5, 6$:

\[
B_3(\tau) = \frac{1}{2}\tau - \frac{3}{2}\tau^2 + \tau^3,
\]
\[
B_5(\tau) = -\frac{1}{6}\tau + \frac{5}{3}\tau^3 - \frac{5}{2}\tau^4 + \tau^5,
\]
\[
B_7(\tau) = \frac{1}{6}\tau - \frac{7}{6}\tau^3 + \frac{7}{2}\tau^5 - \frac{7}{2}\tau^6 + \tau^7,
\]
\[
B_9(\tau) = -\frac{3}{10}\tau + 2\tau^3 - \frac{21}{5}\tau^5 + 6\tau^7 - \frac{9}{2}\tau^8 + \tau^9,
\]
\[
B_{11}(\tau) = \frac{5}{6}\tau - \frac{11}{2}\tau^3 + 11\tau^5 - 11\tau^7 + \frac{55}{6}\tau^9 - \frac{11}{2}\tau^{10} + \tau^{11},
\]
\[
B_{13}(\tau) = \frac{691}{210}\tau^7 + \frac{65}{3}\tau^3 - \frac{429}{10}\tau^5 + \frac{286}{7}\tau^7 - \frac{143}{6}\tau^9 + 13\tau^{11} - \frac{13}{2}\tau^{12} + \tau^{13}.
\]

If the Eq. (5.70) and Eq. (5.71) are inserted in Eq. (5.67) then it holds:

\[
d(\tau) = r(\tau) - \tilde{r}(\tau) = \sum_{\nu=1}^{\infty} d_{\nu}\sin(\nu\pi\tau) = \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau) +
\]

\[
+ \beta \frac{2}{\pi^{2j+2}} \sum_{\nu=1}^{\infty} \left( \frac{1}{\nu^{2j+2}} \int_{\tau'=0}^{1} d_{2j+2}(\tau') \sin(\nu\pi\tau') d\tau' \sin(\nu\pi\tau) \right),
\]

(5.74)

with the coefficients of the Euler polynomials,

\[
e_{2j} := \frac{1}{2(2j)!} \left( d_{2j}(1) + d_{2j}(0) \right),
\]

(5.75)

and the coefficients of the Bernoulli polynomials,

\[
b_{2j+1} := \frac{1}{(2j+1)!} \left( d_{2j}(1) - d_{2j}(0) \right).
\]

(5.76)

The partial integration refers to the normalized time variable $\tau$. If the real time $t$ is used then it holds $d\tau/dt = 1/T$ with the arc length $T = t_B - t_A$ and the relation $\tau = (t - t_A)/T$ with $t \in [t_A, t_B]$,

\[
d^{(2j)} = \frac{d^{2j} d(t)}{dt^{2j}} = \frac{d^{2j} d(\tau)}{d\tau^{2j}} \left( \frac{d\tau}{dt} \right)^{2j} = d^{(2j)} T^{2j},
\]

(5.77)
and analogously the coefficients of the Bernoulli polynomials
\[ b_{2j+1} = \frac{1}{(2j+1)!} \left( d^{(2j)}(1) - d^{(2j)}(0) \right) = \frac{T^{2j}}{(2j)!} \left( d^{(2j)}(t_B) - d^{(2j)}(t_A) \right) = T^{2j} \tilde{b}_{2j+1}, \]  
(5.80)
with \( \tilde{b}_{2j+1} \) as
\[ \tilde{b}_{2j+1} = \frac{1}{(2j+1)!} \left( d^{(2j)}(t_B) - d^{(2j)}(t_A) \right). \]  
(5.81)

The factors are given according to Eqs. (5.75) and (5.76) with \( J_{max}=6 \) as follows:
\[ b_3 = \frac{1}{6} \left( d^{(2)}(1) - d^{(2)}(0) \right) = \frac{T^2}{6} \left( d^{(2)}(1) - d^{(2)}(0) \right), \]
\[ b_5 = \frac{1}{120} \left( d^{(4)}(1) - d^{(4)}(0) \right) = \frac{T^4}{120} \left( d^{(4)}(1) - d^{(4)}(0) \right), \]
\[ b_7 = \frac{1}{5040} \left( d^{(6)}(1) - d^{(6)}(0) \right) = \frac{T^6}{5040} \left( d^{(6)}(1) - d^{(6)}(0) \right), \]
\[ b_9 = \frac{1}{362880} \left( d^{(8)}(1) - d^{(8)}(0) \right) = \frac{T^8}{362880} \left( d^{(8)}(1) - d^{(8)}(0) \right), \]
\[ b_{11} = \frac{1}{39916800} \left( d^{(10)}(1) - d^{(10)}(0) \right) = \frac{T^{10}}{39916800} \left( d^{(10)}(1) - d^{(10)}(0) \right), \]
\[ b_{13} = \frac{1}{6227020800} \left( d^{(12)}(1) - d^{(12)}(0) \right) = \frac{T^{12}}{6227020800} \left( d^{(12)}(1) - d^{(12)}(0) \right), \] and
\[ e_2 = \frac{1}{4} \left( d^{(2)}(1) + d^{(2)}(0) \right) = \frac{T^2}{4} \left( d^{(2)}(1) + d^{(2)}(0) \right), \]
\[ e_4 = \frac{1}{48} \left( d^{(4)}(1) + d^{(4)}(0) \right) = \frac{T^4}{48} \left( d^{(4)}(1) + d^{(4)}(0) \right), \]
\[ e_6 = \frac{1}{1440} \left( d^{(6)}(1) + d^{(6)}(0) \right) = \frac{T^6}{1440} \left( d^{(6)}(1) + d^{(6)}(0) \right), \]
\[ e_8 = \frac{1}{80640} \left( d^{(8)}(1) + d^{(8)}(0) \right) = \frac{T^8}{80640} \left( d^{(8)}(1) + d^{(8)}(0) \right), \]
\[ e_{10} = \frac{1}{7257600} \left( d^{(10)}(1) + d^{(10)}(0) \right) = \frac{T^{10}}{7257600} \left( d^{(10)}(1) + d^{(10)}(0) \right), \]
\[ e_{12} = \frac{1}{958003200} \left( d^{(12)}(1) + d^{(12)}(0) \right) = \frac{T^{12}}{958003200} \left( d^{(12)}(1) + d^{(12)}(0) \right). \]

The Eqs. (5.78) and (5.80) can be used to derive the time derivatives of the difference function \( d(\tau) \) with respect to the time \( t \) at the boundary epochs \( t_A \) and \( t_B \) of the arc,
\[ d^{(2j)}(t_A) = \frac{2(2j)!e_{2j} - (2j+1)! \tilde{b}_{2j+1}}{2} = \frac{2(2j)!e_{2j} - (2j+1)!b_{2j+1}}{2T^{2j}} = \frac{1}{T^{2j}} d^{(2j)}(\tau = 0), \]  
(5.84)
and
\[ d^{(2j)}(t_B) = \frac{2(2j)!e_{2j} + (2j+1)! \tilde{b}_{2j+1}}{2} = \frac{2(2j)!e_{2j} + (2j+1)!b_{2j+1}}{2T^{2j}} = \frac{1}{T^{2j}} d^{(2j)}(\tau = 1). \]  
(5.85)

In the following, we will use both time variables; the differences can be seen from the functional dependencies either on \( t \) or \( \tau \) and from the boundary epochs \( t_A, t_B \) or \( \tau = 0, \tau = 1 \), used. The Euler polynomial coefficients are written \( e_{2j} \) or \( e_{2j} \) and \( \tilde{b}_{2j+1} \) or \( b_{2j+1} \), whether they refer to the real time or the normalized one. The derivatives of the difference function \( d \) with respect to the real time is written as \( d^{(2j)} \) and with respect to the normalized time as \( d^{(2j)} \).
The sine coefficients can be written as
\[ d_{2\nu} = \sum_{j=1}^{J} \frac{2(-1)^{j+1}(2j+1)!}{(2\nu\pi)^{2J+1}} b_{2j+1} + \beta \frac{2}{(2\nu\pi)^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}_{\tau'} \sin 2\nu\pi \tau' \, d\tau', \quad (5.86) \]
and
\[ d_{2\nu-1} = \sum_{j=1}^{J} \frac{4(-1)^{j}(2j)!}{(2\nu-1)^{2J+1}+1\pi^{2j+1}} e_{2j} + \beta \frac{2}{(2\nu-1)^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}_{\tau'} \sin(2\nu-1)\pi \tau' \, d\tau'. \quad (5.87) \]

If we consider the asymptotic behaviors of the integrals in the Eqs. (5.74), (5.86) and (5.87),
\[ \left| \frac{2}{(\nu\pi)^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}_{\tau'} \sin \nu\pi \tau' \, d\tau' \right| = O \left( \frac{1}{\nu^{J+1}} \right), \quad (5.88) \]
then it holds for a sufficient large \( \nu \geq N \)
\[ \left| \frac{2}{(\nu\pi)^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}_{\tau'} \sin \nu\pi \tau' \, d\tau' \right| < \delta, \quad (5.89) \]
with an error \( \delta \) for the coefficients \( d_{\nu,i} \) with \( i = 1, 2, 3 \).

5.3.1.2 Finite Series and Remainder Terms

If the series in Eq. (5.55) and in Eq. (5.74) are summed up to infinity then we can write:
\[ r(\tau) = \bar{r}(\tau) + \sum_{\nu=1}^{n} d_{\nu} \sin(\nu\pi \tau), \quad (5.90) \]
or
\[ d_{\nu}^{F}(\tau) \equiv d_{\nu}^{F}(\tau), \quad (5.91) \]
with
\[ d_{\nu}^{F}(\tau) := \sum_{\nu=1}^{n} d_{\nu} \sin(\nu\pi \tau), \quad (5.92) \]
and
\[ d_{\nu}^{F}(\tau) := \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau). \quad (5.93) \]

If we restrict the upper summation indices to final numbers then we have to take into account remainder functions as follows,
\[ d_{\nu}^{F}(\tau) = d_{\nu}^{F}(\tau) + R_{F}(\tau) = r(\tau) - \bar{r}(\tau) = d_{\nu}^{F}(\tau) + R_{P}(\tau) = d_{\nu}^{F}(\tau). \quad (5.94) \]
The remainder term \( R_{F}(\tau) \) is simply the rest of the Fourier series from index \( n+1 \) to infinity,
\[ R_{F}(\tau) = \sum_{\nu=n+1}^{\infty} d_{\nu} \sin(\nu\pi \tau), \quad (5.95) \]
while the remainder term $\mathbf{R}_p(\tau)$ contains the Euler and Bernoulli polynomials from index $J + 1$ to infinity or high order derivatives of the orbit at the boundaries according to Eq. (5.74),

$$
\mathbf{R}_p(\tau) = \sum_{j=J+1}^{\infty} e_{2j} E_{2j}(\tau) + \sum_{j=J+1}^{\infty} b_{2j+1} B_{2j+1}(\tau)
$$

(5.96)

$$
= \frac{\beta}{\pi^{2J+2}} \sum_{j=1}^{\infty} \left( \frac{1}{\nu^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(\nu \pi \tau') d\tau' \sin(\nu \pi \tau) \right).
$$

The Fourier series coefficients, here the sine coefficients $\mathbf{d}_{2\nu}$ with even indices ($\nu = 1, 2, 3, \ldots$) can be expressed by the coefficients of the Bernoulli polynomials,

$$
\mathbf{d}_{2\nu} = \sum_{j=1}^{J} \frac{2(-1)^{j+1}(2j+1)!}{(2\nu \pi)^{2j+1}} b_{2j+1},
$$

(5.97)

and the coefficients $\mathbf{d}_{2\nu-1}$ with odd indices ($\nu = 1, 2, 3, \ldots$) by the coefficients of the Euler polynomials,

$$
\mathbf{d}_{2\nu-1} = \sum_{j=1}^{\infty} \frac{4(-1)^{j}(2j)!}{(2\nu-1)^{2j+1} \pi^{2j+1}} e_{2j}.
$$

(5.98)

If these series are restricted by an upper index $J$ then the remainder terms $\mathbf{R}_{2\nu}$ and $\mathbf{R}_{2\nu-1}$ of the series

$$
\mathbf{d}_{2\nu} = \sum_{j=1}^{J} \frac{2(-1)^{j+1}(2j+1)!}{(2\nu \pi)^{2j+1}} b_{2j+1} + \mathbf{R}_{2\nu},
$$

(5.99)

and

$$
\mathbf{d}_{2\nu-1} = \sum_{j=1}^{J} \frac{4(-1)^{j}(2j)!}{(2\nu-1)^{2j+1} \pi^{2j+1}} e_{2j} + \mathbf{R}_{2\nu-1}.
$$

(5.100)

can be written according to Eq. (5.86),

$$
\mathbf{R}_{2\nu} = \sum_{j=J+1}^{\infty} \frac{2(-1)^{j+1}(2j+1)!}{(2\nu \pi)^{2j+1}} b_{2j+1} = \frac{\beta}{(2\nu \pi)^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(2\nu \pi \tau') d\tau',
$$

(5.101)

and Eq. (5.87), respectively,

$$
\mathbf{R}_{2\nu-1} = \sum_{j=J+1}^{\infty} \frac{4(-1)^{j}(2j)!}{(2\nu-1)^{2j+1} \pi^{2j+1}} e_{2j} = \frac{\beta}{(2\nu-1)^{2J+2} \pi^{2J+2}} \int_{\tau'=0}^{1} d^{[2J+2]}(\tau') \sin(2\nu-1) \pi \tau' d\tau'.
$$

(5.102)

Because of the fact that there exists a functional dependency between the coefficients $\mathbf{d}_\nu$ of the Fourier series and the coefficients $e_{2j}$ and $b_{2j+1}$ of the Euler- and Bernoulli polynomials the latter ones can be restricted also by the dynamical model, because the Fourier coefficients must fulfill the conditions either according to Eq. (5.13),

$$
\mathbf{d}_\nu \equiv \mathbf{c}_\nu = -\frac{2T^2}{\nu^2 \pi^2} \int_{\tau'=0}^{1} \sin(\nu \pi \tau') \mathbf{a}(\tau'; \mathbf{r}, \dot{\mathbf{r}}) d\tau',
$$

or to Eq. (5.23)

$$
\mathbf{d}_\nu \equiv \mathbf{c}_\nu = -\frac{2T^2}{\nu^2 \pi^2 - \mu^2} \int_{\tau'=0}^{1} \sin(\nu \pi \tau') \mathbf{a}^H(\tau'; \mathbf{r}, \dot{\mathbf{r}}) d\tau'.
$$
or according to Eq. (5.38), respectively,
\[ d_\nu \equiv c_\nu = \tilde{c}_\nu + x_\nu, \]
with Eq. (5.26)
\[ \tilde{c}_\nu = -\frac{2T^2}{\nu^2 \pi^2} \int_{\tau' = 0}^{1} \sin(\nu \pi \tau') \Delta \mathbf{a}(\tau'; \tilde{r}, \dot{\tilde{r}}) d\tau', \]
and Eq. (5.34)
\[ x_\nu = -\frac{2T^2}{\nu^2 \pi^2} \int_{\tau' = 0}^{1} \sin(\nu \pi \tau') \Delta \mathbf{a}(\tau'; \tilde{r} + x, \dot{\tilde{r}} + \dot{x}) d\tau'. \]

Besides this functional dependency of the Fourier coefficients on the dynamical force model, the coefficients of the Euler polynomials are also connected to the dynamical model by the temporal derivatives of the orbit at the boundaries according to Eq. (5.78),
\[ e_{2j} := \frac{T^{2j}}{2(2j)!} \left( \mathbf{d}^{(2j)}(t_B) + \mathbf{d}^{(2j)}(t_A) \right), \]
and the coefficients of the Bernoulli polynomials according to Eq. (5.80),
\[ b_{2j+1} := \frac{T^{2j}}{(2j + 1)!} \left( \mathbf{d}^{(2j)}(t_B) - \mathbf{d}^{(2j)}(t_A) \right). \]

Therefore, both representations of the satellite’s orbit, either in form of the Fourier series or as a series in terms of Euler- and Bernoulli polynomials can be interpreted as a kinematical approximation for the orbit or as solution to the boundary value problem realizing the basis for a dynamical orbit determination.

5.3.2 Determination of the Euler and Bernoulli Polynomial Coefficients

5.3.2.1 Space Domain Representation of the Euler and Bernoulli Polynomial Coefficients

The approximation of the satellite’s arc based on Euler and Bernoulli polynomials reads according to Eq. (5.74) by disregarding the remainder term \( R_P(\tau) \) according to Eq. (5.96),
\[ \mathbf{d}(\tau) = \sum_{\nu=1}^{\infty} d_\nu \sin(\nu \pi \tau) \approx \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau). \]  

In the following, we will write the two sums at the right hand side in matrix notation,
\[ \mathbf{d}(\tau) \approx \sum_{j=1}^{J} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J} b_{2j+1} B_{2j+1}(\tau) = \mathbf{e}^T \mathbf{E} + \mathbf{b}^T \mathbf{B}. \]

The matrices \( \mathbf{E} \) and \( \mathbf{B} \) are composed by the polynomials according to Eqs. (5.72) and (5.73), if we select the upper index \( J_{max} = 6 \), resulting in an upper polynomial degree of 13 without restricting the generality. An upper polynomial degree of 13 means more precisely, degree 12 for the Euler polynomials and degree 13 for the Bernoulli polynomials. The matrices will be labelled accordingly; because of the fact that an Euler polynomial of degree \( 2j \) and a Bernoulli polynomial of degree \( 2j+1 \) belong together as a pair, we will select the upper most degree of such a pair of polynomials as \( \mathbf{E}^{13} \) and \( \mathbf{B}^{13} \). Furthermore, these polynomials are
written as matrix products of the coefficients and the increasing powers of the basic (normalized) time as shown in the following.

\[
E^{13} = E_c^{13} t^{13} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 5 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
17 & 0 & -28 & 0 & 14 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-155 & 0 & 255 & 0 & -126 & 0 & 30 & 0 & -5 & 1 & 0 & 0 & 0 & 0 \\
2073 & 0 & -3410 & 0 & 1683 & 0 & -396 & 0 & 55 & 0 & -6 & 1 & 0 & 0
\end{pmatrix}, \quad (5.105)
\]

and

\[
B^{13} = B_c^{13} t^{13} = \begin{pmatrix}
\frac{1}{5} & -\frac{3}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{5} & 0 & \frac{7}{2} & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10} & 0 & -\frac{7}{6} & 0 & \frac{7}{2} & -\frac{7}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{20} & 0 & 2 & 0 & -\frac{21}{4} & 0 & 6 & -\frac{9}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{210} & 0 & -\frac{11}{5} & 0 & 11 & 0 & -11 & 0 & \frac{55}{6} & -\frac{11}{2} & 1 & 0 & 0 & 0 \\
-\frac{1}{63} & 0 & \frac{5}{3} & 0 & -\frac{429}{10} & 0 & \frac{286}{7} & 0 & -\frac{143}{6} & 0 & 13 & -\frac{13}{2} & 1
\end{pmatrix}.
\]

The matrix of the coefficients of the Euler polynomials reads

\[
e^{T}_{12} = (e_2 e_4 e_6 e_{10} e_{12}),
\]

and the matrix of the coefficients of the Bernoulli polynomials, respectively

\[
b^{T}_{13} = (b_3 b_5 b_7 b_{11} b_{13}),
\]

so that Eq. (5.103) reads with the upper summation index \(J_{\text{max}}=6\)

\[
d(\tau) \approx \sum_{j=1}^{J_{\text{max}}=6} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J_{\text{max}}=6} b_{2j+1} B_{2j+1}(\tau) = e^{T}_{12} E^{13} + b^{T}_{13} B^{13}, \quad (5.109)
\]

or in rearranged form

\[
d(\tau) \approx \sum_{j=1}^{J_{\text{max}}=6} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J_{\text{max}}=6} b_{2j+1} B_{2j+1}(\tau) = (e^{T}_{12} E^{13} + b^{T}_{13} B^{13}) t^{13} = (e^{T}_{12} b^{T}_{13}) \begin{pmatrix} E^{13} \\ B^{13} \end{pmatrix} t^{13}. \quad (5.110)
\]

If we skip the maximal index of the Euler-Bernoulli polynomials and write Eq. (5.110) as follows,

\[
d(\tau) \approx p^{T} p = t^{T} p^{T} p, \quad (5.111)
\]
with
\[ \mathbf{p} := \begin{pmatrix} e_{12} \\ b_{13} \end{pmatrix}, \quad \mathbf{P} := \begin{pmatrix} E^{13} \\ B^{13} \end{pmatrix}, \quad \mathbf{t} \equiv \mathbf{t}^{13} = \begin{pmatrix} \tau \\ \vdots \\ \tau^{13} \end{pmatrix}. \]

(5.112)

If the observational functionals \( \mathbf{d}(\tau_k) \) are available at a sufficient number \( K \) of epochs \( \tau_k \),
\[ \mathbf{d} := \begin{pmatrix} d(\tau_1) \\ \vdots \\ d(\tau_K) \end{pmatrix}, \]
then the system of linear observation equations reads,
\[ \mathbf{d} = \mathbf{T}^T \mathbf{P}^T \mathbf{p}, \]
(5.114)

with the matrix of epochs
\[ \mathbf{T} = \begin{pmatrix} \mathbf{t}_1 & \cdots & \mathbf{t}_K \end{pmatrix} = \begin{pmatrix} \tau_1 & \cdots & \tau_K \\ \vdots & \ddots & \vdots \\ \tau^{13}_1 & \cdots & \tau^{13}_K \end{pmatrix}, \]
(5.115)
or in detail,
\[ \begin{pmatrix} d(\tau_1) \\ \vdots \\ d(\tau_K) \end{pmatrix} \approx \begin{pmatrix} \tau_1 & \cdots & \tau^{13}_1 \\ \vdots & \ddots & \vdots \\ \tau_K & \cdots & \tau^{13}_K \end{pmatrix} \begin{pmatrix} (E^{13})^T \\ (B^{13})^T \end{pmatrix} \begin{pmatrix} e_{12} \\ b_{13} \end{pmatrix}. \]
(5.116)

The coefficients of the Euler and Bernoulli polynomials can be determined by a least squares solution as follows,
\[ \mathbf{p} = (\mathbf{T} \mathbf{T}^T)^{-1} \mathbf{P}^T \mathbf{d}, \]
(5.117)

with the matrix
\[ \mathbf{T} \mathbf{T}^T = \begin{pmatrix} \sum_{k=1}^K (\tau_k)^2 & \cdots & \sum_{k=1}^K \tau_k \tau^{13}_k \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^K \tau^{13}_k \tau_k & \cdots & \sum_{k=1}^K (\tau^{13}_k)^2 \end{pmatrix}, \]
(5.118)

and in detail, respectively,
\[ \begin{pmatrix} e_{12} \\ b_{13} \end{pmatrix} = \left( \begin{pmatrix} (E^{13})^T \\ (B^{13})^T \end{pmatrix} \begin{pmatrix} \sum_{k=1}^K (\tau_k)^2 & \cdots & \sum_{k=1}^K \tau_k \tau^{13}_k \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^K \tau^{13}_k \tau_k & \cdots & \sum_{k=1}^K (\tau^{13}_k)^2 \end{pmatrix} \begin{pmatrix} d(\tau_1) \\ \vdots \\ d(\tau_K) \end{pmatrix} \right)^{-1} \begin{pmatrix} (E^{13})^T \\ (B^{13})^T \end{pmatrix} \begin{pmatrix} d(\tau_1) \\ \vdots \\ d(\tau_K) \end{pmatrix}. \]
(5.119)
5. Representation of Short Arcs

5.3.2.2 Spectral Domain Representation of the Euler and Bernoulli Polynomial Coefficients

If the polynomial coefficients are determined within a least squares adjustment process of the GNSS observations, then the sine coefficients can be computed based on Eq. (5.40),

$$d_\nu = 2 \int_{\tau'=0}^{1} d(\tau') \sin(\nu \pi \tau') \, d\tau',$$

and because of the identity, Eq. (5.42), we can write by inserting e.g. for the case given by Eq. (5.13),

$$d_\nu \equiv c_\nu = -\frac{2T^2}{\nu^2 \pi^2} \int_{\tau'=0}^{1} \sin(\nu \pi \tau') \, d\tau',$$

(5.120)

which interprets these coefficients directly in dependency of the force function model \(a(\tau'; r, \dot{r})\). On the other hand, the coefficients \(d_\nu\) can be determined by the approximate orbit representation, Eq. (5.109),

$$d_\nu \approx 2 \int_{\tau'=0}^{1} \left( \sum_{j=1}^{J_{\text{max}}=6} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{J_{\text{max}}=6} b_{2j+1} B_{2j+1}(\tau) \right) \sin(\nu \pi \tau') \, d\tau'.$$

(5.121)

This equation can be written in matrix form with Eq. (5.110) as following

$$d_\nu \approx 2 \int_{\tau'=0}^{1} \left( e_{12} E_{13} + b_{13} B_{13} \right) t^{13} \sin(\nu \pi \tau') \, d\tau' = \left( e_{12} E_{13} + b_{13} B_{13} \right) I^{13}_{\nu}.$$

(5.122)

The coefficients can be determined by the matrix relation,

$$d_n^T \approx \left( e_{12} E_{13} + b_{13} B_{13} \right) N^{13}_{n},$$

(5.123)

with

$$d_n^T = \begin{pmatrix} d_1 & \cdots & d_\nu & \cdots & d_n \end{pmatrix},$$

(5.124)

and

$$N^{13}_{n} = \begin{pmatrix} I^{13}_1 & \cdots & I^{13}_\nu & \cdots & I^{13}_n \end{pmatrix} = \begin{pmatrix} I^1_1 & \cdots & I^1_\nu & \cdots & I^1_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I^\nu_1 & \cdots & I^\nu_\nu & \cdots & I^\nu_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I^n_1 & \cdots & I^n_\nu & \cdots & I^n_n \end{pmatrix},$$

(5.125)

with the integrals

$$I^{\nu}_n := 2 \int_{\tau'=0}^{1} \tau^{\nu k} \sin(\nu \pi \tau') \, d\tau'.$$

(5.126)

The formula,

$$d_n^T \approx \left( e_{12} E_{13} + b_{13} B_{13} \right) N^{13}_{n} = \left( e_{12} E_{13} + b_{13} B_{13} \right) \left( E^{13}_{13} B^{13}_{13} \right) N^{14}_{n},$$

(5.127)

represents conditions for the Euler polynomials coefficients \(e\) and the Bernoulli polynomial coefficients \(b\) to satisfy the force function \(a(\tau'; r, \dot{r})\) contained in \(d_n\).
To show the structure of these condition equations, we will give an example with the upper limit 5 (J=2) instead of 13. In this case the equation reads as follows:

\[
\begin{pmatrix}
  d_1 & \cdots & d_n
\end{pmatrix} \approx \begin{pmatrix}
  -1 & 1 & 0 & 0 & 0 \\
  1 & 0 & -2 & 1 & 0
\end{pmatrix}
+ \begin{pmatrix}
  b_3 & b_5
\end{pmatrix}
\begin{pmatrix}
  \frac{1}{6} & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & \frac{5}{2} & 1
\end{pmatrix}.
\]

(5.128)

It reads for the sine coefficients \( d_\nu, \nu = 1, \cdots, n, \)

\[
d_\nu = -e_2 I_\nu + e_4 I_\nu^1 + e_2 I_\nu^2 - 2e_4 I_\nu^3 + e_4 I_\nu^4 +
\frac{1}{3}b_3 I_\nu^1 - \frac{1}{6}b_5 I_\nu^2 - \frac{1}{2}b_3 I_\nu^3 + b_3 I_\nu^4 + \frac{3}{4}b_5 I_\nu^5 - \frac{3}{5}b_5 I_\nu^5 + b_5 I_\nu^5.
\]

(5.129)

If sufficient condition equations according to e.g. Eq. (5.120) are introduced into the observation equations to derive the Euler polynomials coefficients \( e \) and the Bernoulli polynomial coefficients \( b \) then a purely dynamic orbit will result. If only selected conditions with specific indices \( \nu \) are introduced then the orbit will be of reduced-kinematical character with a dynamic information.

The Eq. (5.127) can be written in a compact form if we skip the maximal indices of the Euler-Bernoulli polynomials and the matrix of integrals \( N \) as follows,

\[
d_n \approx p^T P N = N^T P^T p.
\]

(5.130)

with

\[
p := \begin{pmatrix}
e_{12} \\
b_{13}
\end{pmatrix}, \quad P := \begin{pmatrix}E^{13}_c & B^{13}_c \end{pmatrix}, \quad N \equiv N^{13}_n.
\]

(5.131)

Eq. (5.130) reads in detail,

\[
\begin{pmatrix}
d_1 \\
\vdots \\
d_n
\end{pmatrix} \approx \begin{pmatrix}
I_1^1 & \cdots & I_1^{13} \\
\vdots & \cdots & \vdots \\
I_n^1 & \cdots & I_n^{13}
\end{pmatrix}
\begin{pmatrix}
(E^{13}_c)^T & (B^{13}_c)^T
\end{pmatrix}
\begin{pmatrix}
e_{12} \\
b_{13}
\end{pmatrix}.
\]

(5.132)

It should be pointed out that the approximate sign "\( \approx \)" holds for the sine coefficients of the real orbit as given by Eq. (5.66) or Eqs. (5.68) and (5.69). They could be determined by Eqs. (5.97) and (5.98) or by Eqs. (5.99) and (5.100). The sign "\( \approx \)" has to be replaced by the equals sign "\( = \)" in Eq. (5.132), if the sine coefficients \( d_\nu \) are understood as the coefficients of the Fourier series of the Euler-Bernoulli polynomials \( d_\nu^J(\tau) \), in the present case for \( J_{max}=6 \). These coefficients can be computed rigorously by Eq. (5.132) or numerically by a discrete Fourier analysis procedure as given, e.g. in Sec. 5.2.4, (see also Sec. 5.4.2.2).

The coefficients of the Euler and Bernoulli polynomials can be determined by a least squares approximation in case of \( n \geq p \) as follows,

\[
p = (P N N^T P^T)^{-1} P N d_n,
\]

(5.133)

with the matrices

\[
N = \begin{pmatrix}
I_1^1 & \cdots & I_n^1 \\
\vdots & \cdots & \vdots \\
I_1^{13} & \cdots & I_n^{13}
\end{pmatrix}, \quad
N^T = \begin{pmatrix}
\sum_{\nu=1}^n (I_1^\nu)^2 & \cdots & \sum_{\nu=1}^n I_1^1 I_1^{13} \\
\vdots & \cdots & \vdots \\
\sum_{\nu=1}^n I_1^{13} I_1^1 & \cdots & \sum_{\nu=1}^n (I_1^{13})^2
\end{pmatrix}.
\]

(5.134)
and in detail, respectively,

$$
\begin{pmatrix}
\mathbf{e}_{12} \\
\mathbf{b}_{13}
\end{pmatrix} =
\begin{pmatrix}
\mathbf{E}^{13}_c & \mathbf{B}^{13}_c \\
\sum_{\nu=1}^{n} I_{\nu}^1 I_{\nu}^{13} & \vdots & \vdots & \sum_{\nu=1}^{n} I_{\nu}^1 I_{\nu}^{13}
\end{pmatrix}^{-1}
\begin{pmatrix}
\mathbf{d}_1 \\
\vdots \\
\mathbf{d}_n
\end{pmatrix}.
$$

(5.135)

It shall be pointed out that the coefficients $\mathbf{d}_\nu$ can be determined either by Eq. (5.40) or by Eq. (5.120). For numerical tests refer to Sec. 5.4.

5.3.2.3 Numerical Quadrature

For the determination of the integrals

$$
I_{\nu}^k := 2 S_{\nu}^k = 2 \int_{\tau=0}^{1} \tau^k \sin(\nu \pi \tau) \, d\tau,
$$

(5.136)

we apply a recursion formula similarly to that which has been proposed in (ILK 1976). It can be derived easily by applying a partial integration to Eq. (5.136):

$$
S_{\nu}^k = \int_{\tau=0}^{1} \tau^k \sin(\nu \pi \tau) \, d\tau = -\frac{\tau^k}{\nu \pi} \cos(\nu \pi \tau) \bigg|_{\tau=0}^{1} + \frac{k}{\nu \pi} \int_{\tau=0}^{1} \tau^{k-1} \cos(\nu \pi \tau) \, d\tau.
$$

(5.137)

If we insert the following abbreviations into Eq. (5.137),

$$
C_{\nu}^k := \int_{\tau=0}^{1} \tau^k \cos(\nu \pi \tau) \, d\tau,
$$

(5.138)

then the integrals $S_{\nu}^k$ can be determine as follows

$$
S_{\nu}^k = -\frac{(-1)^{\nu}}{\nu \pi} + \frac{k}{\nu \pi} C_{\nu}^{k-1}.
$$

(5.139)

A similar relation can be derived for the integrals $C_{\nu}^k$,

$$
C_{\nu}^k = \int_{\tau=0}^{1} \tau^k \cos(\nu \pi \tau) \, d\tau = \frac{\tau^k}{\nu \pi} \sin(\nu \pi \tau) \bigg|_{\tau=0}^{1} - \frac{k}{\nu \pi} \int_{\tau=0}^{1} \tau^{k-1} \sin(\nu \pi \tau) \, d\tau = -\frac{k}{\nu \pi} S_{\nu}^{k-1}.
$$

(5.140)

The recursion formula will be applied for a fixed index $\nu$ and the set of powers of $\tau^k$ with $k = 0, ..., J + 1$. The initial functionals are simply

$$
C_{\nu}^{0} = C_{\nu}^{k=0} = \int_{\tau=0}^{1} \tau^0 \cos(\nu \pi \tau) \, d\tau = 0,
$$

(5.141)

and

$$
S_{\nu}^{0} = S_{\nu}^{k=0} = \int_{\tau=0}^{1} \tau^0 \sin(\nu \pi \tau) \, d\tau = -\frac{1}{\nu \pi} \cos(\nu \pi \tau) \bigg|_{\tau=0}^{1} = \frac{1}{\nu \pi} (1 - (-1)^{\nu}),
$$

(5.142)
so that the following recursion scheme follows,
\[ S_k^\nu = -\frac{(-1)^\nu}{\nu\pi} + \frac{k}{\nu\pi} C_k^{\nu-1}, \quad C_k^\nu = -\frac{k}{\nu\pi} S_k^{\nu-1}. \] (5.143)

The final results can be derived by a multiplication by the factor 2,
\[ I_k^\nu := 2S_k^\nu. \] (5.144)

It should be pointed out that this integrals can be determined once for all and can be used for all orbit determination cases.

Unfortunately, these simple and fast formula show a very critical numerical behavior for high indices \( k \) and small indices \( \nu \). This can be seen if the Eq. (5.139) are written in closed form,
\[ C_k^\nu = \sin \nu\pi F_k^1 + \cos \nu\pi F_k^2 + F_k^3, \] (5.145)
and
\[ S_k^\nu = \sin \nu\pi F_k^2 - \cos \nu\pi F_k^1 + F_k^4, \] (5.146)

with
\[ F_k^1 = \sum_{i=0}^{[k/2]} \frac{(-1)^i}{(\nu\pi)^{2i+1}} \frac{k!}{(k-2i)!}, \] (5.147)
\[ F_k^2 = \sum_{i=0}^{[k-1/2]} \frac{(-1)^i}{(\nu\pi)^{2(2i+1)}} \frac{k!}{(k-2i-1)!}, \] (5.148)
\[ F_k^3 = \frac{1}{2} \frac{(-1)^k}{\nu\pi} \frac{k+1}{k} (\nu\pi)^{k+1}, \] (5.149)
\[ F_k^4 = \frac{(-1)^k + 1}{2} \frac{k!}{(\nu\pi)^{k+1}}. \] (5.150)

These functions contain very large terms with alternating signs with the consequence that leading numbers cancel, because the absolute values of the integrals are small, fulfilling the inequalities,
\[ |C_k^\nu| \leq \frac{1}{k+1}, \quad |S_k^\nu| \leq \frac{1}{k+1}. \] (5.151)

The integrals \( S_k^\nu \) can be determined alternatively by a series expansion of the sine function and a subsequent analytical integration and an insertion of the integration limits
\[ S_k^\nu = I_k^\nu = \int_{\tau=0}^1 \tau^k \sin(\nu\pi\tau) \, d\tau = \sum_{i=0}^{\infty} \frac{(-1)^i (\nu\pi)^{2i+1}}{(2i+1)! (2i+2+k)}. \] (5.152)

These series show a fast convergence with small index \( \nu \) and large index \( k \).

5.3.3 Temporal Derivatives of the Orbit Function

5.3.3.1 Velocity

In the following, we will use the hybrid version of the orbit representation, that means, we will represent the orbit by three terms, a first one represents a reference motion \( \bar{r}(\tau) \), a second one represents the Euler-Bernoulli polynomials \( d^J_P(\tau) \) and a third one the Fourier series \( d^F_P(\tau) \). The last two difference orbit representations
can be used alternatively or as combination representation. If we set one of the indices $J$ or $n$ zero then we get the specific alternatives as discussed in the last sections. The velocities along the orbit can be derived by a differentiation of the orbit function, Eq. (5.90),

$$\mathbf{r}(t) = \mathbf{r}(\tau) \approx \ddot{\mathbf{r}}(\tau) + \mathbf{d}_p^J(\tau) + \mathbf{R}_{PF}(\tau) \approx \dot{\mathbf{r}}(\tau) + \mathbf{d}_p^J(\tau) + \mathbf{d}_p^0(\tau),$$  \hspace{1cm} (5.153)

where the normalized time $\tau$ is related to the real time $t$ according to Eq. (5.3),

$$\tau = \frac{t - t_A}{T} \quad \text{with} \quad T = t_B - t_A, \quad t \in [t_A, t_B],$$

with respect to the time

$$\frac{d\mathbf{r}(\tau)}{dt} = \frac{d\mathbf{r}(\tau)}{dT} + \frac{d\mathbf{d}_p^J(\tau)}{dt} + \frac{d\mathbf{d}_p^0(\tau)}{dt},$$  \hspace{1cm} (5.154)

and with $\mu$ according to Eq. (5.15),

$$\frac{d\dot{\mathbf{r}}(\tau)}{dt} = \frac{\mu}{T \sin \mu} (-\cos \mu (1 - \tau)\mathbf{r}_A + \cos \mu \tau \mathbf{r}_B),$$  \hspace{1cm} (5.155)

and the terms

$$\frac{d\mathbf{d}_p^J(\tau)}{dt} = \frac{1}{T} \left( \sum_{j=1}^{J} 2je_{2j}E_{2j-1}(\tau) + \sum_{j=1}^{J} (2j + 1)b_{2j+1} B_{2j}(\tau) \right),$$  \hspace{1cm} (5.156)

$$\frac{d\mathbf{d}_p^0(\tau)}{dt} = \frac{1}{T} \sum_{\nu=1}^{n} d_{\nu} \nu \pi \cos (\nu \pi \tau).$$  \hspace{1cm} (5.157)

Then the velocities can be determined by the function

$$\frac{d\mathbf{r}(\tau)}{dt} = \frac{\mu}{T \sin \mu} (-\cos \mu (1 - \tau)\mathbf{r}_A + \cos \mu \tau \mathbf{r}_B) +$$

$$+ \frac{1}{T} \left( \sum_{j=1}^{J} 2je_{2j}E_{2j-1}(\tau) + \sum_{j=1}^{J} (2j + 1)b_{2j+1} B_{2j}(\tau) \right) + \frac{1}{T} \sum_{\nu=1}^{n} d_{\nu} \nu \pi \cos (\nu \pi \tau).$$  \hspace{1cm} (5.158)

If the upper index $n$ is not sufficiently high then Gibbs’ effects can cause errors especially at the boundaries of the arcs. If the force function $\mathbf{a}(\tau')$ is available, then it is preferable to determine first the velocity at the time $t_A$ according to (ILK 1976),

$$\dot{\mathbf{r}}_A = \frac{1}{T} (\mathbf{r}_B - \mathbf{r}_A) - T \int_{\tau'=0}^{1} (1 - \tau') \mathbf{a}(\tau') \, d\tau',$$

and then the velocity at an arbitrary time instant $t$ by an integration as follows,

$$\dot{\mathbf{r}}(t) = \dot{\mathbf{r}}_A + (t - t_A) \int_{\tau'=0}^{1} \mathbf{a}(\tau') \, d\tau',$$

with

$$\tau' = \frac{t' - t_A}{T} \in [0, 1], \quad t, t' \in [t_A, t_B],$$  \hspace{1cm} (5.161)

or, by transformation of the integration interval,

$$\dot{\mathbf{r}}(\tau) = \dot{\mathbf{r}}_A + T \int_{\tau'=0}^{\tau} \mathbf{a}(\tau') \, d\tau',$$

with

$$\tau' = \frac{t' - t_A}{T} \in [0, 1], \quad \tau = \frac{t - t_A}{T} \in [0, 1].$$  \hspace{1cm} (5.163)
5.3. Series of Euler–and Bernoulli Polynomials

5.3.3.2 Acceleration

The force function corresponds to the second derivative of the orbit with respect to the time according to Eq. (5.1) with Eq. (5.6)

\[ L(r(t)) = \frac{d}{dt} \ddot{r}(t) = a(\tau). \]  

(5.164)

It holds for the second derivatives

\[ \frac{d\ddot{r}(\tau)}{d\tau} = \frac{d\dot{\vec{r}}(\tau)}{d\tau} + \frac{d\hat{\mathcal{J}}_p(\tau)}{d\tau} + \frac{d\hat{\mathcal{B}}_p(\tau)}{d\tau}, \]  

(5.165)

with

\[ \frac{d\dot{\vec{r}}(\tau)}{d\tau} = \frac{\mu^2}{T^2 \sin \mu} (-\sin \mu (1 - \tau) r_A - \sin \mu \tau r_B), \]  

(5.166)

\[ \frac{d\hat{\mathcal{J}}_p(\tau)}{d\tau} = \frac{1}{T^2} \left( \sum_{j=1}^{J} 2j(2j-1)e_{2j} E_{2j-2}(\tau) + \sum_{j=1}^{J} 2j(2j+1)b_{2j+1} B_{2j-1}(\tau) \right), \]  

(5.167)

\[ \frac{d\hat{\mathcal{B}}_p(\tau)}{d\tau} = -\frac{1}{T^2} \sum_{\nu=1}^{n} d_{\nu} (\nu \pi)^2 \sin(\nu \pi \tau). \]  

(5.168)

The accelerations can be determined by

\[ \frac{d\ddot{r}(\tau)}{d\tau} = \frac{\mu^2}{T^2 \sin \mu} (-\sin \mu (1 - \tau) r_A - \sin \mu \tau r_B) + \]  

\[ + \frac{1}{T^2} \left( \sum_{j=1}^{J} 2j(2j-1)e_{2j} E_{2j-2}(\tau) + \sum_{j=1}^{J} 2j(2j+1)b_{2j+1} B_{2j-1}(\tau) \right) - \frac{1}{T^2} \sum_{\nu=1}^{n} d_{\nu} (\nu \pi)^2 \sin(\nu \pi \tau), \]  

(5.169)

which corresponds to the force function \( a(\tau) \).

5.3.3.3 Check

If this formula is inserted into Eq. (5.159) instead of the force function \( \ddot{r}(\tau) = a(\tau) \) and with it in (5.160) then we arrive again at Eq. (5.158). This shall be demonstrated for the velocity at the time \( t_A \). It holds

\[ \dot{r}_A = \frac{1}{T} (r_B - r_A) - T \int_{\tau'=0}^{1} (1 - \tau') \left( \frac{d\ddot{r}(\tau')}{d\tau'} + \frac{d\hat{\mathcal{J}}_p(\tau')}{d\tau'} + \frac{d\hat{\mathcal{B}}_p(\tau')}{d\tau'} \right) d\tau', \]

(5.170)

with the integrals

\[ \int_{\tau'=0}^{1} (1 - \tau') \frac{d\ddot{r}(\tau')}{d\tau'} d\tau' = \]  

\[ = -\frac{\mu^2}{T^2 \sin \mu} \left( r_A \int_{\tau'=0}^{1} (1 - \tau') \sin \mu (1 - \tau') d\tau' + r_B \int_{\tau'=0}^{1} (1 - \tau') \sin \mu \tau' d\tau' \right) = \]  

(5.171)

\[ = -\frac{1}{T^2 \sin \mu} (r_A (\sin \mu - \mu \cos \mu) + r_B (\mu - \sin \mu)), \]
and

\[
\int_{\tau' = 0}^{1} (1 - \tau') \frac{dA_{j}^{1}(\tau')}{dt} \, d\tau' =
\]

\[
= \frac{1}{T^2} \int_{\tau' = 0}^{1} \left( \sum_{j=1}^{J} 2j(2j + 1)\mathbf{e}_{2j} E_{2j, -2}(\tau') + \sum_{j=1}^{J} 2j(2j + 1)\mathbf{b}_{2j+1} B_{2j+1}(\tau') \right) (1 - \tau') \, d\tau'
\]

(5.172)

\[
= \frac{1}{T^2} \left( \sum_{j=1}^{J} 2j(2j - 1)\mathbf{e}_{2j} \int_{\tau' = 0}^{1} (1 - \tau') \, E_{2j, -2}(\tau') \, d\tau' + \sum_{j=1}^{J} 2j(2j + 1)\mathbf{b}_{2j+1} \int_{\tau' = 0}^{1} (1 - \tau') \, B_{2j+1}(\tau') \, d\tau' \right)
\]

\[
= -\frac{1}{T^2} \left( \sum_{j=1}^{J} (E_{2j}(\tau' = 0) - E_{2j}(\tau' = 1) + 2jE_{2j, -1}(\tau' = 0))\mathbf{e}_{2j} + \sum_{j=1}^{J} (2j + 1)B_{2j}(\tau' = 1)\mathbf{b}_{2j+1} \right),
\]

as well as with

\[
\int_{\tau' = 0}^{1} (1 - \tau') \frac{dA_{j}^{1}(\tau')}{dt} \, d\tau' = -\frac{1}{T^2} \sum_{\nu=1}^{n} \mathbf{d}_{\nu} (\nu\pi)^2 \int_{\tau' = 0}^{1} (1 - \tau') \, \sin(\nu\pi\tau') \, d\tau'
\]

\[
= -\frac{1}{T^2} \sum_{\nu=1}^{n} \mathbf{d}_{\nu} (\nu\pi - \sin(\nu\pi)).
\]

(5.173)

The velocity at the starting point can be determined by the formula,

\[
\dot{\mathbf{r}}_{A} = \frac{\mu}{T \sin \mu} (-\cos \mu \mathbf{r}_{A} + \mathbf{r}_{B}) +
\]

(5.174)

\[
+ \frac{1}{T} \left( \sum_{j=1}^{J} 2jE_{2j, -1}(\tau = 0)\mathbf{e}_{2j} + \sum_{j=1}^{J} (2j + 1)B_{2j}(\tau = 0)\mathbf{b}_{2j+1} \right) + \frac{1}{T} \sum_{\nu=1}^{n} \mathbf{d}_{\nu} \nu \pi.
\]

This equation coincides with Eq. (5.158) for the time \( \tau = 0 \).

## 5.4 Numerical Verifications

In this section, various numerical tests are performed which shall demonstrate the approximation characteristics and the numerical properties of the formulae derived so far. Furthermore, these tests shall be helpful for the subsequent developments to construct procedures for the kinematical and reduced kinematical orbit determination. The computations are performed based on noise-free simulated data sets; details will be given in the different sub-sections. In a first subsection, the approximation properties of the Fourier series, the series in terms of Euler and Bernoulli polynomials as well as combinations of them are tested. The latter modification is the most promising and will be used later on for the construction of a proper orbit determination technique. In a second subsection, the dynamical restrictions on the various parameters, which model the kinematical orbits are tested. This is important in so far as these restrictions force some kinematical orbit parameters to fulfill a selected dynamical model. Selected dynamical restrictions will be used to construct "reduced" kinematical orbit determination modifications.

### 5.4.1 Approximation Tests

#### 5.4.1.1 The Approximation by a Finite Fourier Series

If the sine series is limited by a finite upper index \( n \) according to Eq. (5.94), then it holds,

\[
\mathbf{r}(\tau) = \bar{\mathbf{r}}(\tau) + \mathbf{d}_{F}^{\infty}(\tau) = \bar{\mathbf{r}}(\tau) + \mathbf{d}_{F}(\tau) + \mathbf{R}_{F}(\tau),
\]

(5.175)
with the approximation function

$$d^p_F(\tau) = \sum_{\nu=1}^{n} d_\nu \sin(\nu \pi \tau),$$

(5.176)

and the remainder term \( R_F(\tau) \),

$$R_F(\tau) = \sum_{\nu=n+1}^{\infty} d_\nu \sin(\nu \pi \tau).$$

(5.177)

The function \( d^p_F(\tau) \) itself as well as the remainder function \( R_F(\tau) \) depend on the reference motions \( \bar{r}(\tau) \). It can be modelled by \( \bar{r}(\tau) \) either according to Eq. (5.10) as a straight line (Fig. 5.1),

$$\bar{r}(\tau) = (1 - \tau) r_A + \tau r_B,$$

or according to Eq. (5.20) as an ellipse with the center at the origin of the reference system (Fig. 5.2),

$$\bar{r}(\tau) = \frac{\sin \mu (1 - \tau)}{\sin \mu} r_A + \frac{\sin \mu \tau}{\sin \mu} r_B,$$

or according to Eq. (5.37),

$$\bar{r}(\tau) = \bar{\bar{r}}(\tau) + \bar{x}(\tau),$$

as a dynamical reference orbit \( \bar{r}(\tau) \) until to a specified degree of the gravity field model in addition to a linear combination of the boundary values of the difference function \( \bar{x}(\tau) \), according to Eq. (5.33) (Fig. 5.3),

$$\bar{x}(\tau) = (1 - \tau) x_A + \tau x_B.$$

To show the approximation characteristics in case of the various choices of reference motions, a 30 minute arc has been simulated based on a gravity field model up to a maximal spherical harmonic degree of \( N_F = 300 \). In case of a reference orbit as reference motion a gravity field model up to a spherical harmonic degree of \( N_R = 30 \) has been selected. The remainder functions will depend on the approximation functions and the various modes of the reference motions \( \bar{r}(\tau) \). They are shown for different maximal upper indices of the Fourier series. If the maximal index \( n \) is too small then a pronounced Gibb’s effect will occur. The Figs. 5.4, 5.6 and 5.8 show the spectra of the various reference motions divided in various spectral bounds for a better visibility and Figs. 5.5, 5.7 and 5.9 the corresponding remainder functions for different upper series indices \( n \).

It becomes clear that the kind of reference motion \( \bar{r}(\tau) \) is important for the size of the amplitudes but also for the size of the remainder functions. A reference motion based on a gravity field of a maximal upper degree and order of \( N_R = 30 \) models the (pseudo) real orbit with an accuracy of less than a meter. If, in addition, the sine series is restricted by an upper index of 50 then the remainder function deviates from the real orbit by less than a mm. In summary, we can state that the accuracy becomes much better by using as reference motion a dynamically determined arc than a straight line or an elliptic motion. On the other hand, one has to keep in mind that the determination of a dynamical reference arc is more costly than the use of an elliptic reference. Therefore, the latter one might be an acceptable compromise.

### 5.4.1.2 The Approximation by a Finite Series of Euler and Bernoulli Polynomials

If the solution series in terms of Euler and Bernoulli polynomials are limited by a finite upper index \( J \) according to Eq. (5.94), then it holds

$$r(\tau) = \bar{r}(\tau) + d^p_F(\tau) = \bar{r}(\tau) + d^p_F(\tau) + R_F(\tau),$$

(5.178)
5. Representation of Short Arcs

Figure 5.4: Spectrum of $d_J^n(\tau)$ based on $\tilde{f}(\tau)$ selected as reference motion in the straight line mode for an Earth gravity field model up to degree and order $N_F = 300$.

with the approximation function

$$d_J^n(\tau) = \sum_{j=1}^J e_{2j} E_{2j}(\tau) + \sum_{j=1}^J b_{2j+1} B_{2j+1}(\tau),$$

(5.179)

and the remainder term $R_P(\tau)$,

$$R_P(\tau) = \sum_{j=J+1}^\infty e_{2j} E_{2j}(\tau) + \sum_{j=J+1}^\infty b_{2j+1} B_{2j+1}(\tau).$$

(5.180)

Analogously to the last section, the function $d_J^n(\tau)$ itself as well as the remainder function $R_P(\tau)$ depend on the reference motion $\tilde{f}(\tau)$, either according to Eq. (5.10) as a straight line, according to Eq. (5.20) as an ellipse or according to Eq. (5.37) as a dynamical reference orbit $\tilde{f}(\tau)$.

To show the approximation characteristics in case of the various choices of reference motions, again a 30 minute arc has been simulated based on a gravity field model up to a maximal spherical harmonic degree of $N_F = 300$. In case of a reference orbit as reference motion a gravity field model up to a spherical harmonic degree of $N_R = 30$ has been selected. The remainder functions will depend on the approximation functions and the various modes of the reference motion $f(\tau)$. In the following, they are shown for different maximal
upper indices of the series in terms of Euler-Bernoulli polynomials. It became clear in Sec. 5.3 that the Euler polynomials of degree $2j$ and the Bernoulli polynomials of degree $2j + 1$ belong together for a specific index $j$. Figs. 5.11, 5.13 and 5.15 show the remainder functions for various maximal series indices $J = 1, 2, 3, 4, 5$ based on the reference motions (i.e. straight line, ellipse and reference orbit degree and order 30 modes) as discussed before. Figs. 5.10, 5.12 and 5.14 show the series contributions for the indices $j = 1, 2, 3, 4, 5$ of an Euler-Bernoulli polynomial approximation with a maximal index $J_{\text{max}} = 5$ for the different reference motions. It is interesting to note that the approximation quality increases with increasing upper index $J$. Nevertheless, the series contributions of the separate polynomial degrees $J = 1, 2, 3, 4$ and 5 to the series $d_{J_{\text{max}}=5}^T(\tau)$ do not become smaller. The reason is not completely clear; it seems that the convergence of the Euler-Bernoulli polynomials require a large upper index $J$ until the convergence can be directly seen.

5.4.1.3 The Approximation by a Combination of Fourier Series and Series of Euler-Bernoulli Polynomials

In the last section, it is demonstrated that the upper index $J$ of the series in terms of Euler-Bernoulli polynomials should be very high to achieve an approximation accuracy at the sub millimeter level. But it should be pointed out once again that a pure representation by a series of Euler-Bernoulli polynomials is not in the focus of this approach. Rather a fit of a series in terms of Euler-Bernoulli polynomials to the geometrically determined arc at a reasonable upper degree (e.g. $J_{\text{max}}=4$) should guarantee a sufficient
confident determination of the Euler-Bernoulli coefficients, corresponding to sufficiently precise arc derivatives at the arc boundary epochs. In that case, the residual sine series should show a fast convergence and low residuals of the combined series when compared to the true ephemerides. Detailed investigations to this facts are performed in Sec. 5.4.2.2. To demonstrate the numerical characteristics of the combined series the simulation cases as used in the above section have been considered in the straight line, elliptic and reference orbit mode. Then a series of Euler-Bernoulli polynomials up to degree $J_{\text{max}}=4$ has been fitted to the difference function $d(\tau)$ and subsequently a sine series has been determined for the Euler-Bernoulli remainder function. Fig. 5.16 shows the various spectral bands of the amplitudes for the "straight line" mode. Fig. 5.17 shows the residuals of the combined series "Euler-Bernoulli polynomials" plus "sine series" up to different upper summation limits. The figure shows from top to bottom the residuals of the approximation cases, where the sine series are limited by the upper indices 20, 30, 50, 70, 90, 100, 150 and 178 and finally by the index 179 which represents the interpolation case; the residuals are very small and caused only by rounding errors. Similar cases are shown in Figs. 5.18 and 5.19 for the "elliptic mode" and Figs. 5.20 and 5.21 for the "reference orbit mode". The combination of a series in terms of Euler-Bernoulli polynomials up to degree $J_{\text{max}}=4$ and a series of sine functions shows a very good approximation quality for all modes of reference motions. This is especially the case for the straight line mode but also for the ellipse mode. This is a consequence of the similar convergence characteristics of the spectra of the real orbit and the approximation by Euler-Bernoulli polynomials (refer to Sec. 5.4.2.2). The case of a dynamical reference orbit up to degree

**Figure 5.8:** Spectrum of $d_F^p(\tau)$ based on $\bar{f}(\tau)$ selected as reference motion in the reference orbit up to degree and order 30 mode for an Earth gravity field model up to degree and order $N_F=300$.

**Figure 5.9:** Remainder function $R_F(\tau)$ based on $\bar{f}(\tau)$ selected as reference motion in the reference orbit up to degree and order 30 mode for different Fourier indices $n$ for an Earth gravity field model up to degree and order $N_F=300$. 
5.4. Numerical Verifications

$N_R = 30$ shows similar results as in the case of a pure sine series approximation. In that case, the reference orbit obviously shows a similar approximation quality as the Euler-Bernoulli polynomial series up to degree $J_{\text{max}} = 4$ so that the approximation of the remainder function in terms of sine series is approximately identical. Because of the fact that the computation of the dynamical reference orbit is much more costly than the use of an ellipse as reference motion the latter one should be preferred.

5.4.2 Dynamical Restrictions on the Kinematical Orbit Parameters

5.4.2.1 Euler and Bernoulli Polynomial Coefficients and the Derivatives of the Position Function at the Orbit Boundaries

As outlined in sections 5.2 and 5.3, there is a functional dependency between the coefficients of the Euler and Bernoulli polynomials and the derivatives of the difference function $\mathbf{d}(t)$ at the boundaries of a satellite arc. This is a first dynamical restriction on the coefficients of the Euler-Bernoulli polynomials if these derivatives are derived from the force function acting on the satellite. This is obvious for the second derivatives which
Figure 5.12: Series contributions of the separate degrees $j$=1,2,3,4 and 5 to the sum $d_j\tau$ based on $\vec r(\tau)$ (reference motion: ellipse mode) for an Earth gravity field model up to degree and order $N_F=300$.

correspond to the force functions at the boundaries. Higher derivatives can be derived from the force function model as well. The relations are given by Eq. (5.75) for the coefficients of the Euler polynomials,

$$e_{2j} = \frac{1}{2(2j)!} \left( d^{(2j)}(1) + d^{(2j)}(0) \right),$$

and by Eq. (5.76) for the coefficients of the Bernoulli polynomials,

$$b_{2j+1} = \frac{1}{(2j+1)!} \left( d^{(2j)}(1) - d^{(2j)}(0) \right).$$

From these formulae it is easy to derive the derivatives of $d(t)$, $d^{(2j)}(t)$ with $j = 1, 2, 3, 4, ...$, at $t_A$ of the arc, Eq. (5.84),

$$d^{(2j)}(t_A) = \frac{2(2j)! e_{2j} - (2j + 1)! b_{2j+1}}{2T^{2j}},$$

and at $t_B$, Eq. (5.85),

$$d^{(2j)}(t_B) = \frac{2(2j)! e_{2j} + (2j + 1)! b_{2j+1}}{2T^{2j}}.$$
5.4. Numerical Verifications

The Euler and Bernoulli polynomials of degree $2j$ and $2j + 1$, respectively, are closely related to the time derivatives of the orbit function at the boundaries of the arc. This is the reason that these polynomials have to be considered together and should be characterized by the index $j$. Unfortunately, it is not easy to determine analytically higher derivatives in a simple straight forward way, except as mentioned in the case of the first derivatives, which correspond to the velocity and in the case of the second derivatives, the accelerations, which coincide with the specific force function at the arc boundaries. Numerically, one could approximate the differential quotients by the difference quotients. Because of subtractive cancellation and the resulting loss of significance the computations require a very high number of digits. Here the computations are performed by Martin Ettl (Fundamental Station Wettzell) who developed a computer code based on the LiDIA library with 200 digits after the comma (personal communication). This is certainly no possibility in real case applications but a way to check the numerical characteristics in this research. Therefore, in the following, we will determine the coefficients $e_{2j}$ and $b_{2j+1}$ by a least squares fit of the Euler-Bernoulli polynomials to a specific orbit limited by an upper index $J_{\text{max}} = 6$ according to Eq. (5.109),

$$d_0^P(\tau) := \sum_{j=1}^{6} e_{2j} E_{2j}(\tau) + \sum_{j=1}^{6} b_{2j+1} B_{2j+1}(\tau) \approx \bar{r}(\tau) - \overline{r}(\tau).$$

(5.181)
Figure 5.16: Spectrum of the Euler Bernoulli remainder function (reference motion: straight line mode, $J_{\text{max}} = 4$).

Figure 5.17: Residuals for the combined series Euler Bernoulli polynomials and sine series with various upper limits (reference motion: straight line mode, $J_{\text{max}} = 4$).
Figure 5.18: Spectrum of the Euler Bernoulli remainder function (reference motion: ellipse mode, $J_{\text{max}} = 4$).

Figure 5.19: Residuals for the combined series Euler Bernoulli polynomials and sine series with various upper limits (reference motion: ellipse mode, $J_{\text{max}} = 4$).
Figure 5.20: Spectrum of the Euler Bernoulli remainder function (reference motion: dynamical reference orbit up to degree and order 30 ($N_{R}=30$), $J_{\text{max}}=4$).

Figure 5.21: Residuals for the combined series Euler Bernoulli polynomials and sine series with various upper limits (reference motion: dynamical reference orbit up to degree and order 30 ($N_{R}=30$), $J_{\text{max}}=4$).