Photo-Realistic Rendering of Fiber Assemblies

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Dipl.-Inform. Arno Zinke

Bonn, 3. August 2007

Universität Bonn
Institut für Informatik II
Römerstraße 164, D-53117 Bonn
Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms Universität Bonn

1. Referent: Prof. Dr. Andreas Weber
2. Referent: Prof. Dr. Reinhard Klein
3. Referent: Prof. Dr.-Ing. Marcus Magnor


Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn http://hss.ulb.uni-bonn.de/diss_online elektronisch publiziert.

Erscheinungsjahr: 2008
ABSTRACT

In this thesis we introduce a novel uniform formalism for light scattering from filaments, the Bidirectional Fiber Scattering Distribution Function (BFSDF). Similar to the role of the Bidirectional Surface Scattering Reflectance Distribution Function (BSSRDF) for surfaces, the BFSDF can be seen as a general approach for describing light scattering from filaments. Based on this theoretical foundation, approximations for various levels of abstraction are derived allowing for efficient and accurate rendering of fiber assemblies, such as hair or fur. In this context novel rendering techniques accounting for all prominent effects of local and global illumination are presented. Moreover, physically-based analytical BFSDF models for human hair and other kinds of fibers are derived. Finally, using the model for human hair we make a first step towards image-based BFSDF reconstruction, where optical properties of a single strand are estimated from "synthetic photographs" (renderings) a full hairstyle.
CONTENTS

1. Introduction .................................................. 1
   1.1 Organization ............................................. 2

2. Optical Properties of Fibers ............................... 3
   2.1 Filaments and Fibers .................................. 3
       2.1.1 Optical Material Properties of Fibers .......... 3
       2.1.2 Human Head Hair ................................ 6
       2.1.3 Wet Hair ......................................... 10

3. Light Scattering from Filaments ........................... 15
   3.1 Introduction ............................................. 15
   3.2 Related Work ........................................... 17
   3.3 Motivation and General Assumptions ................. 19
   3.4 Notations ................................................. 20
   3.5 Bidirectional Fiber Scattering Distribution Function — BFSDF . 24
   3.6 Dielectric Fibers ....................................... 26
       3.6.1 Dielectric Cylinder .............................. 30
   3.7 Far-Field Approximation and BCSDF .................... 33
   3.8 Further Special Cases ................................ 37
       3.8.1 Near-Field Scattering with Constant Incident Lighting . 38
       3.8.2 Curve Scattering with Locally Varying Incident Lighting . 38
   3.9 Previous Fiber Scattering Models ....................... 39
       3.9.1 Kajiya & Kay’s Model ............................ 39
       3.9.2 The Model of Marschner et al. (2003) ........... 40
   3.10 Examples of Further Analytic Solutions and Approximations .... 43
       3.10.1 Opaque Circular Symmetric Fibers ............ 44
3.10.2 A Practical Parametric Near-Field Shading Model for Dielectric Fibers with Elliptical Cross Section .......................... 46
3.11 Extending the BFSDF ........................................... 49
3.12 Conclusion ......................................................... 51
3.13 Results .......................................................... 52

4. Rendering of Fiber Assemblies ................................. 59
4.1 Introduction ....................................................... 59
4.2 Geometrical Representation of a Fiber ........................ 60
4.3 BFSDF and BCSDF Rendering ................................ 62
  4.3.1 Towards Efficient BFSDF and BCSDF Rendering ....... 62
4.4 Local and Global Illumination .................................. 64
4.5 Global Illumination for Fiber Assemblies ......................... 65
  4.5.1 Related Work .................................................. 66
  4.5.2 Overview ...................................................... 68
4.6 Monte Carlo Particle Tracing ..................................... 69
4.7 BFSDF Monte Carlo Path tracing ................................ 70
4.8 Efficient Ray-Based Global Illumination ......................... 71
  4.8.1 Introduction and Related Work ............................ 71
  4.8.2 Overview .................................................... 75
  4.8.3 Building a Photon Map ..................................... 76
  4.8.4 Lookup in a Photon Map .................................... 77
  4.8.5 Sampling Error .............................................. 77
  4.8.6 Radiance Estimation ........................................ 79
  4.8.7 Results and Conclusion .................................... 79
4.9 Efficient Ray-Based Global Illumination for Fiber Assemblies . 82
  4.9.1 Overview .................................................... 83
  4.9.2 Visibility Check and LOD .................................. 84
  4.9.3 Particle Tracing Pass ....................................... 85
  4.9.4 Radiance Reconstruction ................................... 86
  4.9.5 Approximating the BFSDF Rendering Integral .......... 87
  4.9.6 Radiance Estimation ....................................... 89
  4.9.7 Implementation and Results ............................... 91
4.9.8 Comparison to Other Photon Mapping Based Approaches 92
4.10 Physically Plausible Approximation of Multiple Fiber Scattering for Hair ......................................................... 95
  4.10.1 Basic Ideas and Assumptions ........................................ 97
  4.10.2 Approximation of Global Backscattering by a Local BCSDF 99
  4.10.3 Backscattering from a Cluster ....................................... 102
  4.10.4 Approximation of Global Forward Scattering ................. 111
  4.10.5 Combining the Models for Global Backward and Forward Scattering ......................................................... 113
  4.10.6 Modifications and Details ........................................... 116
  4.10.7 Rendering .................................................................. 117
  4.10.8 Results ..................................................................... 119
  4.10.9 Conclusion and Future Work ........................................ 122
4.11 Pseudo Codes for Sec. 4.9 and Sec. 4.10 .......................... 123
4.12 Results ........................................................................ 130
  4.12.1 Comparison of Results: Path Tracing vs Approximations 130
  4.12.2 Results of the Ray-Based Approximation ..................... 137
  4.12.3 Results of the “Double Scattering” Approximation ....... 140
5. Towards Inverse BCSDF Rendering of Human Hair ................ 151
  5.1 Related Work .................................................................. 152
  5.2 Key Ideas of our Approach .............................................. 152
  5.3 Discussion of Results and Future Work ........................... 156
6. Conclusion and Future Work ........................................... 163
  6.1 Conclusion .................................................................... 163
  6.2 Outlook ......................................................................... 164
7. Appendix .......................................................................... 165
  7.1 Basics of Optics .............................................................. 165
    7.1.1 Law of Reflection ...................................................... 165
    7.1.2 Snell’s Law .............................................................. 165
    7.1.3 Fresnel Equations ..................................................... 166
  7.2 Bravais’s Law ................................................................. 167
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.3</td>
<td>Transformation of the BFSDF Rendering Integral</td>
<td>171</td>
</tr>
<tr>
<td>7.4</td>
<td>Derivation of Marschner et al.</td>
<td>173</td>
</tr>
<tr>
<td>7.4.1</td>
<td>R-component</td>
<td>175</td>
</tr>
<tr>
<td>7.4.2</td>
<td>TT-component</td>
<td>176</td>
</tr>
<tr>
<td>7.4.3</td>
<td>TRT-component</td>
<td>177</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

Modeling of fiber assemblies is essential to computer graphics for a variety of applications. A very prominent example is the modeling of hair and fur for virtual characters. In this context diverse problems have to be solved to create believable photo-realistic characters. This includes the generation of convincing hair-styles, the simulation of hair dynamics as well as hair rendering [62, 14, 55].

All these problems draw their complexity mainly from the huge number of interacting individual hair strands. Another issue is the long but comparatively thin structure of a strand combined with its highly nonlinear deformation under external load. This makes a modeling very sensitive to the choice of computation techniques and may cause sampling and stability artifacts. Finally, light scattering from hair fibers is highly anisotropic and inter-reflections play an important role for the right visual perception of the color. Hence, not only local but also global scattering effects have to be taken into account.

Besides of its application to virtual characters, accurate optical simulations of hair are gaining relevance also in the cosmetic industry. Here, the interest focuses mainly on the prediction of the hair color under different illuminations and for different hairstyles.

Another active area of research which is closely related to the modeling of fiber assemblies is the simulation of woven tissue and knitwear [6, 9]. However, in contrast to hair modeling the possible deformation of a tissue is highly restricted due to the constraints caused by the regular structure of the weaving pattern. Therefore the approaches used to model the dynamics of tissue and knitwear substantially differ from hair.

This thesis focuses on efficient and physically accurate visualization of large fiber assemblies, mainly in the context of hair rendering. Based on a novel framework for light scattering from a single fiber efficient rendering techniques for vari-
ous levels of abstraction are presented and other aspects of hair and fiber rendering are discussed.

1.1 Organization

In chapter 2, we will give a brief overview over optically relevant properties of fibers in general. In this context we will also discuss light scattering from human hair fibers including the influence of water. Chapter 3 describes a novel scattering formalism for light scattering from filaments. Based on this framework new rendering techniques are presented in chapter 4. Finally, an approach for image-based reconstruction of hair scattering functions is sketched in chapter 5. We close the thesis with a conclusion and proposals for future research activities.
2. OPTICAL PROPERTIES OF FIBERS

2.1 Filaments and Fibers

In the following the terms fiber and filament will be used interchangeably. They always stand for a class of materials that are either continuous thin structures or discrete elongated pieces, such as: hair or fur fibers, yarns, wires ropes or grass (Fig. 2.1). To analyze scattering from fibers their structure and the impact on scattering has to be analyzed. In this context it is often useful to decouple local from global properties. For example the way a fiber is bent or warped can be separated from the local geometry of its cross section.

A fiber may be a composite material like yarns and human hair or can consist of only one matter as in the case of optical fibers or metal wires. In principle it is possible to estimate scattering from a fiber from its geometric shape and the optical properties of its components. However, this requires a deep understanding of structural and optical interrelationships. In the next section only a brief overview over some of the relevant optical properties is given. For further reading on scattering in matter we refer to the book written by van de Hulst [60].

2.1.1 Optical Material Properties of Fibers

There is a variety of optical properties describing light scattering in matter. However, often only four wavelength dependant properties are needed to estimate the scattering distribution: the scattering coefficient, the phase function, the index of refraction (or refractive index) and the absorption coefficient.

The scattering coefficient is a factor that expresses the attenuation of radiant energy per unit length caused by scattering during its passage through a medium. Depending on the medium the scattering of light may be either isotropic
or anisotropic. In this context the phase function is a probability density function describing the angular distribution of volumetric light scattering.

For dielectric and conducting media the dimensionless index of refraction plays an important role. It is closely related to the phase velocity of light in matter and affects the scattering paths of light as well as the transmittance and the reflectance of a material (see also App. 7.1.2, App. 7.1.3). Note that reflection or refraction can in principle change the polarization of light. However, if not stated differently unpolarized light is assumed for all further considerations.

If light enters a dielectric it gets attenuated. The attenuation per unit length is given by another optical property, which is called the absorption coefficient.

By simulating light transport at a fiber, it is basically possible to render photorealistic images of fiber based geometries for arbitrary lighting.

Since a very relevant application of fiber rendering is hair rendering, the next section will explain phenomenological effects dominating the appearance of human hair. Moreover, a simple physically-based scattering model accounting for these effects is introduced.
2.1. Filaments and Fibers

Fig. 2.1: Examples of fiber based geometries and materials. **Top left:** red silk fabric (http://www.echinatex.net/images/upload/product_200661111835.jpg). **Top right:** cotton grass (http://www.naturschutzstation.de/bilder/pflanzen/wollgrass_steff.jpg). **Bottom left:** human hair. **Bottom right:** a cable drum (http://www.optocore.com/pics/products/Optocable800.jpg).
The average human head has about 100,000 hair strands. Each strand consists of layered keratin structures which can be roughly classified as either cuticle or cortex (see Fig. 2.2). The inner of the hair is called the cortex. It contains a bunch of cell strings and determines most optical properties of a hair strand. Pigment granules that are distributed throughout the hair cortex and which are concentrated in its core (the medulla) give the hair its distinctive coloring. However, the overall hair color may exhibit extreme variations with respect to viewing direction, illumination and the entire hairstyle.

The hair cortex is covered by several layers composed of flat, overlapping cells. This scaly structure is called the cuticle. Due to the overlapped arrangement the surface of the scales exhibit a systematic deviation from the normal of a smooth fiber.

The overall thickness of a human hair is generally between $50 - 100\mu m$ in the European population, but it can average $120\mu m$ in some Asian populations. However, not only the width of a hair fiber may vary, but also the cross section geometry. Asian hair tends to have a nearly circular cross section, whereas African hair exhibits a substantial ellipticity [56]. For optical properties of human hair, such as absorption coefficients and refractive index, we refer to table 3.1 in section 3.10.2.
Fig. 2.3: Blond hair: **Left:** A photograph of blond hair. **Right:** A close-up of the photograph shown left. Note the refractions indicating that light scattering from a hair strand is quite similar to scattering from a dielectric cylinder.

According to Marschner et al. [37] the morphology of the hair provides an adequate explanation for all prominent features of scattering from a single hair strand. Substantial measurements indicated that human hair fibers can be approximated very well by transparent dielectric cylinders with a colored interior and a surface composed of rough, tilted scales. An example, which supports this assumption is shown in Fig. 2.3.

Marschner et al. showed that all relevant features observed during measurements can be explained by the first three orders of scattering from a smooth dielectric cylinder, which contain the biggest fraction of scattered power (see Fig. 2.4). These scattering components are denoted R-, TT- and TRT-component, respectively. The R-component is the direct surface reflection and since no absorption takes place light gets attenuated only by Fresnel reflection (App. 7.1.3). For the two other components light covers some distance inside the fiber, leading to two colored highlights, a strong forward scattering highlight (TT—two times transmitted) which is visible when hair is lit from behind and a backward scattering lobe (TRT—two times transmitted and internally reflected). Some photographs illustrating the appearance of these highlights are shown in Fig. 2.5.
Fig. 2.4: Modeling light scattering from a hair strand based on a dielectric cylinder. **Left:** The first three scattering modes (R, TT, TRT) of a smooth dielectric cylinder. **Right:** Scattering from a hair fiber. Due to the tiled surface scales the highlights are shifted with respect to the perfect specular direction.

Another effect which can be explained by using the cylinder model are "glints". This phenomenon (commonly seen in people of European and African descent, less so in Asians) causes the hair to behave similar to a lens, focusing light in certain directions. As it is shown in Fig. 2.5 (rightmost image) they are an important part of the appearance, and they lend a distinctive texture to the hair.

Generally, human hair fibers can not be treated as perfect-smooth cylinders. Since the hair surface is tiled, the scattering differs from a smooth cylinder: The R mode is displaced towards the root, TT and TRT towards the tip. Moreover spatial and angular blurring occurs (scattering coefficients > 0) and the cross section may vary in shape. Also structural variations, such as local variation of the pigment density, influence scattering. However, in the following it is assumed that such effects can be handled by slight modifications of the original dielectric cylinder model. For a detailed description of this model as well as a more formal discussion of scattering from dielectric fibers in general, we refer to sections 3.6, 3.9.2 and 3.10.2.
Fig. 2.5: Highlights. **Left:** Hair is lit from behind: colored TT-component. **Middle:** A single front light: white R-component, colored TRT-component. **Right:** Texture due to glints (from [37]).
2. Optical Properties of Fibers

2.1.3 Wet Hair

Fig. 2.6: Comparison of a brown hairstyle (images from [63]) before (left) and after treatment with water (right). Note the darkening of the overall hair color and the shinier appearance of the wet hair.

When hair is treated with water its appearance changes. Although several publications are addressing this phenomenon the proposed scattering models are purely phenomenological guesses without any further qualitative or quantitative analysis [63, 64, 14]. After discussing the effects occurring for wet hair we will show that they can be modeled by using a simple physically-based approach.

As illustrated in figure 2.6 and figure 2.7 the geometry changes (agglutination) and the hair looks more shinier and darker if hair is treated with water. The increase in reflectivity happens mainly because the dry surface exhibits substantial surface roughness, whereas a liquid layer around the hair leads to a nearly smooth surface. Generally, more roughness gives a wider specular lobe (R-component), thus less prominent intensity peaks. Moreover, the water layer causes additional highlights (denoted R',R'',...) due to inter reflections between the air-liquid and liquid-hair interfaces, further increasing the overall reflectivity (see Fig. 2.8 Left).

The darkening is a combination of several effects. First of all the agglutination of hair strands locally increases the hair density, strengthening the effective attenuation in the hair volume. Moreover— according to [63]— water is absorbed
into the hair fiber further increasing the opacity of a strand. However, also the water layer has a substantial impact on the appearance. It induces stronger internal reflections reducing the intensity of scattering components, since more energy gets absorbed inside the fiber. Because the absorption coefficient is usually a wavelength dependant property this also changes the overall coloring of the internally scattered highlights. Moreover, stronger multiple internal scattering is spreading the energy over area and solid angle. Due to this effect, colored highlights get blurred and become less prominent.

The fiber based effects of darkening and increased shininess can be qualitatively simulated and validated using a rather simple physically based model: A dielectric cylinder with an index of refraction matching the effective index of hair (which is 1.55, c.f. [57]) surrounded by a thin water layer (index of refraction is 1.33). We implemented a particle tracer to estimate the spatial variation of the radiant flux at the surface of the fiber according to Fig. 2.8 Right. Some results illustrating the influence of the water layer are shown in Fig. 2.9 and Fig. 2.10. Note in particular the darkening of the TRT-lobe and the additional highlights. Varying the thickness of the water layer did not qualitatively change these effects.
Fig. 2.8: **Left:** Illustration of a wet hair strand. The reflectivity increases due to multiple reflection highlights (R,R,R′...). Moreover—due to stronger internal reflections—more energy "stays" inside the fiber, causing darkening of the colored highlights. **Right:** Virtual flux measurements using a particle tracing approach for an accurate simulation of scattering from a single fiber (see also Sec. 4.6). A cylindrical collector which is wrapped around the fiber is used to estimate the exitant flux through the surface of the fiber, if a finite surface patch is illuminated from a fixed light direction. The local flux is exemplary sketched for the R- and the TRT-component.
Fig. 2.9: Virtual flux measurement at a dry dielectric fiber with a circular cross section. The fiber consists of dielectric material (absorption: $10 \text{cm}^{-1}$, index of refraction: 1.55) and exhibits surface and volumetric roughness leading to spatial and angular blurring of the first three scattering components (R,TT,TRT). A single surface patch of the fiber is illuminated by directional light (angle of incidence is 30° “from the right”). The flux is estimated as illustrated in Fig. 2.8 and the image shows the total flux at the (unwrapped) cylindrical collector.

Fig. 2.10: Virtual flux measurement at a dielectric fiber (cf. Fig. 2.9) surrounded by a water layer (thickness of the water layer is 10% of the fiber diameter). Note the additional white highlights (compared to Fig. 2.9) and the overall darkening and blurring of the internally scattered components.
2. Optical Properties of Fibers
3. LIGHT SCATTERING FROM FILAMENTS

3.1 Introduction

Light scattering from a single filament is the basis of physically accurate rendering of fiber assemblies. As long as the light transport mechanisms for light scattering are understood and if the internal structure of a filament is perfectly known, rendering a single filament is usually not very challenging. In this case the filament could be basically modeled geometrically by structures having certain optical properties like index of refraction or absorption coefficient. However, for the frequent case of a scene consisting of a huge number of individual fibers (like hair, fur or woven knitwear) this explicit approach becomes impracticable due to its space and time complexity. It is also not clear a priori, how to deal with very complex or unknown scattering mechanisms or how efficient visualizations of fibers based on measurement data could be implemented.

In this section we introduce a novel concept similar to Bidirectional Scattering-Surface Reflectance Distribution Functions (BSSRDF) and Bidirectional Scattering Distribution Functions (BSDF) for surfaces in order to describe the light scattering from a filament. The basic idea is to locally approximate the scattering distribution by a function parameterized at the minimum enclosing cylinder of a straight infinite fiber, which is a first order approximation with respect to the curvature of the filament. We call this approximation a Bidirectional Fiber Scattering Distribution Function (BFSDF) (see section 3.5).

Based on the BFSDF we then derive further less complex scattering functions for different levels of geometric abstraction and for special lighting conditions, cf. Fig. 3.1. With the help of such scattering functions efficient physically based visualizations—adapted to a desired rendering technique or quality—become possible. Furthermore the BFSDF allows for systematic comparisons and classifica-
3. Light Scattering from Filaments

Fig. 3.1: Overview of the BFSDF and its special cases. The effective dimensions of the scattering functions are indicated on the left.

...tions of fibers with respect to their scattering distributions, since it provides a uniform and shape independent radiance parameterization for filaments (see sections 3.6, 3.7 and 3.8). Moreover, we show that the existing models developed for hair rendering [25, 37] can be integrated in our framework, which also gives a basis to discuss their physical plausibility (section 3.9). We furthermore address some special cases where analytical solutions or approximations are available.
3.2 Related Work

Although progress has been made in rendering particular fibers with particular lighting settings no general formulation has been found. As a result some simplistic ad-hoc models were developed, especially in the realm of hair rendering [25, 12, 27]. All these models have no strong physical background and need substantial user interaction to yield physically plausible results. If the viewing or lighting conditions change such models may fail or have to be adopted.

Basically fibers could be modeled by conventional geometrical primitives (e.g. cylinders). Hence, in principal light scattering from filaments could be computed using existing scattering functions for surfaces and participating media. For surfaces the Bidirectional Scattering Distribution Function (BSDF) relates the incoming irradiance at an infinitesimal surface patch to the outgoing radiance from this patch [47]. It is an abstract optical material property which decouples microscopic features influencing the scattering behavior from the macroscopic shape of a surface and implies a constant wave length, a light transport taking zero time and being temporally invariant as well. With the BSDF of a certain material, the local surface normal and the incoming radiance, the outgoing radiance can be reconstructed. This BSDF concept works well for a wide range of materials but has one major drawback: Incoming irradiance contributes only to the outgoing radiance from a patch, if it directly illuminates this patch. Therefore it has to be generalized to take into account light transport inside the material. This generalization is called a Bidirectional Surface Scattering Distribution Function—BSSRDF [47]. Several papers have addressed some special cases mainly in the context of subsurface scattering [23, 41].

Further scattering models like the BTF or even more generalized radiance transfer functions have been introduced to account for self-shadowing and other mesoscopic and macroscopic effects [10]. For rendering of participating media commonly phase functions are used. Phase functions are very similar to the BSDF and describe volumetric scattering at a point in space.

In the realm of physically based fiber rendering Marschner et al. [37] presented an approach specialized in rendering human hair strands. They very briefly introduce a curve scattering function defined with respect to curve intensity (intensity
scattered per unit length of the curve) and curve irradiance (incoming power per unit length). Unfortunately no hint about how it can be derived for other types of fibers than hair is given. Another major restriction of the scattering model is that it describes far-field scattering only. This means that both observer (virtual camera) and light sources have to be distant to the hair fiber. If one of these two assumptions is violated it is very likely that the model introduces substantial bias. In particular for inter-reflections between neighboring fibers near-field effects play a prominent role. Therefore this approach is critical for rendering multiple fiber scattering, which is essential for light colored hair. A further issue is close-ups, since fine scattering detail across the width of a fiber can not be resolved (Fig.3.2).

Another field of application of fiber rendering is the visualization of woven cloth, where optical properties of a single yarn have to be taken into account. A ray-tracing based method for estimating the BSDF for a certain weaving pattern and cylindric fibers was sketched by Volevich et al. [61]. However, since all fibers are modeled explicitly the approach is very computationally costly. Gröller et al. proposed a volumetric approach for modeling knitwear [13]. They first measure the statistical density distribution of the cross section of a single yarn with respect to the arrangement of individual yarn fibers and then translate it along a three dimensional curve to form the entire filament. The results are looking quite impressive, but the question as to deal with yarns with more complex scattering properties is not addressed. A similar idea was presented by [67]. Here all computations base on a structure called lumislice, a light field of a yarn-cross-section. However, the authors do not discuss the problem of computing a physically based light field according to the properties of a single fiber of the yarn. Adabala et al. [1] describe another method for visualizing woven cloth. Due to the limitations of the underlying BRDF (Cook-Torance microfacet BRDF) realistic renderings of yarns with more complex scattering properties are impossible.

In the realm of applied optics, light scattering from straight smooth dielectric fibers with constant and mainly circular or elliptical cross sections were analyzed in a number of publications [3, 2, 4, 44, 39]. Schuh et al. [52] also introduced an approximation which is capable of predicting scattering of electro-magnetic waves from curved fibers. All these approaches are of very high quality, but are either more specialized in predicting the maxima positions of the scattering distri-
 Motivation and General Assumptions

Over the last two decades it turned out that the BRDF/BSDF and the BSSRDF are very successful concepts to describe light scattering from surfaces. Our goal is to find similar approaches for fibers by adopting the basic ideas for surfaces to the realm of fiber optics. The general problem is very similar: How much radiance \( dL_o \) scattered from a single fiber one would observe from an infinitesimal surface patch \( dA_o \) in direction \( \omega_o \), if a surface patch \( dA_i \) is illuminated by an irradiance \( E_i \) from direction \( \omega_i \).

This interrelationship could be basically described by a BSSRDF at the surface of the fiber (Fig. 3.3 Left). Such a BSSRDF would in general depend on macroscopic deformations of the fiber, how the fiber is oriented and warped. As a consequence local (microscopic) fiber properties could not be separated from its global macroscopic geometry. Furthermore, all radiometric quantities must be parameterized with respect to the actual surface of the filament, which may not be well defined—e.g. in the case of a fluffy wool yarn.
Therefore we define a BSSRDF on the local infinite minimum enclosing cylinder rather than on the actual surface of the fiber in order to make the parameterization independent of the fiber’s geometry (Fig. 3.3 Right). Thus the radiance transfer at the fiber is locally approximated by a scattering function on the minimum enclosing cylinder of a straight infinite fiber, which means a first order approximation with respect to the curvature. We call this function a Bidirectional Fiber Scattering Distribution Function (BFSDF). This approximation is accurate, if the influence of curvature to the scattering distribution can be neglected. This is the case if e.g.

- the filament’s curvature is small compared to the radius of its local minimum enclosing cylinder or
- most of the incoming radiance only locally contributes to outgoing radiance.

The latter condition is satisfied for instance by opaque wires, since only reflection occurs and therefore no internal light transport inside the filament takes place. Hence, the scattering is a purely local phenomenon and the curvature of the fiber plays no role at all. The former condition for instance holds for hair and fur. In this case substantial internal light transport takes place, but since the curvature is small compared to the radius and since the light gets attenuated inside the fiber, the differences compared to a straight infinite fiber may be neglected. The higher the curvature and the more internal light transport takes place, the bigger the potential error that is introduced by a BFSDF.

### 3.4 Notations

Before technically defining the BFSDF we will give an overview of our notations. The local filament axis (tangent) is denoted \( \vec{u} \) and we refer to the planes perpendicular to \( \vec{u} \) as normal planes. All local surface normals lie within the local normal plane. All incoming and outgoing radiance is parameterized with respect to the local minimum enclosing cylinder of radius \( r \) oriented along \( \vec{u} \) and is therefore independent of the actual cross section geometry of the filament (Fig. 3.4 Left). In the following we will consider a ray with direction \( \vec{\omega} \) which is intersecting the
Fig. 3.3: Description of scattering at a fiber with a pentagonal cross section by: \textbf{Left:} a BSSRDF at the actual surface of the fiber. \textbf{Right:} a BFSDF at the local minimum enclosing cylinder.

minimum enclosing cylinder at a point \( \tilde{X} \). The infinitesimal surface patch centered around \( \tilde{X} \) will be denoted \( dA \).

We introduce two sets of variables, which are suitable for certain measurement or lighting settings. The first parameter set describes \( \tilde{X} \) by its tangential position \( s \) along the tangent and its azimuthal position \( \xi \) within the normal plane (cylindrical coordinates). The direction \( \tilde{\omega} \) of a ray intersecting the cylinder is then given by two spherical angles \( \alpha \) and \( \beta \) which span a hemisphere at \( \tilde{X} \) oriented along the local surface normal of the enclosing cylinder (Fig. 3.4 Right, Fig. 3.5 Right).

Typically this very intuitive way to parameterize both directions and positions is not very well suited to actual problems. In particular, this is the case for data acquisition and many “real world” lighting conditions. Hence, we introduce a second set of variables according to [37]. This set can be split into two groups, one parameterizing all azimuthal features within the normal plane \((h, \varphi)\) and another parameterizing all longitudinal features along the tangent \((s, \theta)\). The angle \( \theta \) denotes the inclination of \( \tilde{\omega} \) with respect to the normal plane, \( \varphi \) its azimuthal angle with respect to a fixed axis \( \tilde{v} \) and \( h \) an offset position of \( \tilde{X} \) at the minimum enclosing cylinder, measured within the normal plane. The variable \( s \) is the same as introduced before and always means the position along the axis of the minimum enclosing cylinder (Fig. 3.5).

We will now derive some useful transformation formulae for variables of both sets. The projection of the angle of incidence at the enclosing cylinder onto the
Fig. 3.4: **Left:** An "intuitive" variable set for parameterizing incoming and outgoing radiance at the minimum enclosing cylinder. The vector \( \vec{u} \) denotes the local tangent (axis), \( s \) the position along this axis. The angles \( \alpha \) and \( \beta \) span a hemisphere over the surface patch at \( \vec{X} \), with \( \alpha \) being measured with respect to \( \vec{u} \) within the tangent plane and \( \beta \) with respect to the local surface normal \( \vec{n} \). **Right:** Azimuthal variables of the first and the second parameter set. The parameter \( h \) is an offset position at the perimeter. Note that parallel rays have same \( \varphi \) (measured counterclockwise with respect to \( \vec{u} \)) but different \( h \).

The normal plane \( \gamma' \) is given by the following two equations:

\[
\cos \gamma' = \frac{\cos \beta}{\cos \theta} \\
\gamma' = |\arcsin h| \tag{3.1}
\]

Thus for the spherical angle \( \beta \) we have:

\[
\beta = \arccos \left( \cos \theta \sqrt{1 - h^2} \right). \tag{3.3}
\]

The following relation holds between \( \xi \), \( \varphi \) and \( h \), cf. Fig. 3.4 Right:

\[
\xi = \begin{cases} 
\varphi + \arcsin h & 0 \leq \varphi + \arcsin h < 2\pi \\
2\pi + \varphi + \arcsin h & \varphi + \arcsin h < 0 \\
\varphi + \arcsin h - 2\pi & \varphi + \arcsin h \geq 2\pi
\end{cases} \tag{3.4}
\]
Fig. 3.5: **Left:** Parameterization with respect to the normal plane and filament axis (tangent) \( \vec{u} \). The vector \( \vec{ω}' \) denotes the projection of a direction \( \vec{ω} \) onto the normal plane. The angle \( θ \) ranges from \(-\pi/2\) to \(\pi/2\), with \( θ = \pi/2 \) if \( \vec{u} \) and \( \vec{ω} \) are pointing in the same direction. **Right:** Parameterization of the BFSDF.

The spherical coordinate \( α \) equals the angle between the projection of \( \vec{ω} \) onto the local tangent plane and \( \vec{v} \). Since the angle between \( \vec{ω} \) and \( \vec{v} \) is given by \( π/2 − θ \) and the inclination of \( \vec{D} \) with respect to the tangent plane is \( π/2 − β \) one obtains:

\[
\cos α = \frac{\cos (π/2 − θ)}{\cos (π/2 − β)} = \frac{\sin θ}{\sin β} = \frac{\sin θ}{\sqrt{1 − (1 − h^2) \cos^2 θ}}.
\]

Furthermore, the actual angle depends on the sign of \( h \):

\[
α = \begin{cases} 
\arccos \left( \frac{\sin θ}{\sqrt{1 − (1 − h^2) \cos^2 θ}} \right), & h \leq 0, \\
2π − \arccos \left( \frac{\sin θ}{\sqrt{1 − (1 − h^2) \cos^2 θ}} \right), & h > 0.
\end{cases}
\]
3.5 Bidirectional Fiber Scattering Distribution Function — BFSDF

We now technically define the Bidirectional Fiber Scattering Distribution Function (BFSDF) of a fiber. Please note that for further considerations all radiometric quantities at the minimum enclosing cylinder are implicitly transformed by a transformation that unwraps the cylinder to a plane. By applying such a transformation the local surface normals of the cylinder become equally aligned. As a consequence the angles $\alpha$ and $\beta$ can be expressed with respect to a global coordinate system.

The BFSDF relates incident flux $\Phi_i$ at an infinitesimal surface patch $dA_i$ to the outgoing radiance $L_o$ at another position on the minimum enclosing cylinder:

$$f_{\text{BFSDF}}(s_i, \xi_i, \alpha_i, \beta_i, s_o, \xi_o, \alpha_o, \beta_o) := \frac{dL_o(s_i, \xi_i, \alpha_i, \beta_i, s_o, \xi_o, \alpha_o, \beta_o)}{d\Phi_i(s_i, \xi_i, \alpha_i, \beta_i)}.$$  

(3.7)

Using the BFSDF the local orientation of a fiber and the incoming radiance $L_i$, the total outgoing radiance of a particular position can be calculated by integrating the irradiance over all surface patches and all incoming directions as follows:

$$L_o(s_o, \xi_o, \alpha_o, \beta_o) = \int_{-\infty}^{+\infty} \int_{0}^{2\pi} \int_{0}^{\pi/2} f_{\text{BFSDF}}(s_i, \xi_i, \alpha_i, \beta_i, s_o, \xi_o, \alpha_o, \beta_o) L_i(s_i, \xi_i, \alpha_i, \beta_i) \sin \beta_i \cos \beta_i d\beta_i d\alpha_i d\xi_i ds_i.$$  

(3.8)

For a physically based BFSDFs the energy is conserved, thus all light entering the enclosing cylinder is either absorbed or scattered. In case of perfectly circular symmetric filaments the BFSDF in addition satisfies the Helmholtz Reciprocity:

$$f_{\text{BFSDF}}(s_i, \xi_i, \alpha_i, \beta_i, s_o, \xi_o, \alpha_o, \beta_o) = f_{\text{BFSDF}}(s_o, \xi_o, \alpha_o, \beta_o, s_i, \xi_i, \alpha_i, \beta_i).$$  

(3.9)

If the optical properties of a fiber are constant along $s$, then the BFSDF does not depend on $s_i$ and $s_o$ but on the difference $\Delta s := s_o - s_i$. Thus the dimension of
the BFSDF function decreases by one and the scattering integral reduces to the following equation:

\[
L_o(s_o, \xi_o, \alpha_o, \beta_o) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} \int_{0}^{\pi/2} f_{BFSDF}(\Delta s, \xi_i, \alpha_i, \beta_i, \xi_o, \alpha_o, \beta_o) L_i(s_i, \xi_i, \alpha_i, \beta_i) \sin \beta_i \cos \beta_i \, d\beta_i \, d\alpha_i \, d\xi_i \, d\Delta s.
\]

(3.10)

This special case is also an appropriate approximation for the general case if the following condition is satisfied:

- The cross section shape and material properties vary slowly along \( s \) or
- have only very high frequency detail on a scale which does not have to be resolved by the BFSDF.

Good examples for high frequency detail are cuticula tiles of hair fibers or surface roughness due to the microstructure of a filament. Since this assumption is at least locally valid for a wide range of different fibers, we will restrict ourselves to this case in the following.

The BFSDF and its corresponding rendering integral was introduced with respect to the “intuitive variable set” first. We now derive a formulation for the second variable set. Reparameterizing the scattering integral yields:

\[
L_o(s_o, h_o, \varphi_o, \theta_o) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} \int_{0}^{\pi/2} \frac{\partial}{\partial (h_i, \varphi_i, \theta_i)} \left| \begin{array}{c}
\xi_i \\
\alpha_i \\
\beta_i
\end{array} \right| f_{BFSDF}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) L_i(s_i, h_i, \varphi_i, \theta_i) \sqrt{1 - \cos^2 \theta_i (1 - h_i^2)} \sqrt{1 - h_i^2 \cos \theta_i} \\
L_i(s_i, h_i, \varphi_i, \theta_i) \, d\theta_i \, d\varphi_i \, dh_i \, d\Delta s.
\]

(3.11)

The above equation can be finally simplified to the following equation (see also App. 7.3):
3. Light Scattering from Filaments

\[ L_o(s_i, h_i, \varphi_i, \theta_i) = \int_{-\infty}^{+\infty} \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} f_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \cos^2 \theta_i L_i(s_i, h_i, \varphi_i, \theta_i) \, d\theta_i \, d\varphi_i \, dh_i \, d\Delta s. \]  

(3.12)

This form of the scattering integral is especially suitable when having distant lights, i.e. parallel illumination, since parallel rays of light have same \( \varphi_i \) and \( \theta_i \) but different \( h_i \). Consider for example directional lighting, which can be expressed by the radiance distribution \( L(\varphi'_i, \theta'_i, \varphi_i, \theta_i) = E_\perp(\varphi'_i, \theta'_i) \delta(\varphi_i - \varphi'_i) \delta(\theta_i - \theta'_i) / \cos \theta_i \) with normal irradiance \( E_\perp(\varphi'_i, \theta'_i) \). Substituting the radiance in the scattering integral by this distribution yields the following special case:

\[ L_o(h_o, \varphi_o, \theta_o) = E_\perp(\varphi'_i, \theta'_i) \cos \theta'_i \int_{-\infty}^{+\infty} \int_{-1}^{1} f_{\text{BFSDF}}(\Delta s, h_i, \varphi'_i, \theta'_i, h_o, \varphi_o, \theta_o) \, dh_i \, d\Delta s. \]  

(3.13)

All formulae derived in this section hold for a single filament only. But a typical scene consists of a substantial number of densely packed fibers. Since incoming and outgoing radiance was parameterized with respect to the minimum enclosing cylinder, those cylinders must not intersect. Otherwise it can not be guaranteed that the results are still valid (Fig. 3.6 Left). If a cross section shape closely matches the enclosing cylinder the maximum overlap is very small. Hence, computing the light transport at a projection of the overlapping regions onto the enclosing cylinder can be seen as a good approximation of the actual scattering.

### 3.6 Dielectric Fibers

Since most filaments consist of dielectric material it is very instructive to analyze the scattering from dielectric fibers. For perfectly smooth dielectric filaments with arbitrary but constant cross section the dimension of the BFSDF reduces from seven to six. This is a direct consequence of the fact that light entering at a certain
inclination will always exit at the same inclination (according to absolute value), regardless of the sequence of refractions and reflections it undergoes. Thus—neglecting longitudinal positions $s$—all scattered light lies on a cone.

The incident light is reflected or refracted several times before it leaves the fiber. The amount of light being reflected resp. refracted is given by Fresnel’s Law. At each surface interaction step the scattered intensity of a ray decreases—except in case of total reflection. If absorption inside the fiber takes place, then loss of energy is even more prominent. Since an incoming ray of light does not exhibit spatial or angular blurring, the BFSDF is characterized by discrete peaks. Therefore, it makes sense to analyze only those scattering components with respect to their geometry and attenuation which:

- can be measured outside the fiber and
- have an intensity bigger than a certain given threshold.

As a result, the BFSDF can be approximated very well by the distribution of its strongest peaks. Then the effective complexity further reduces to three dimensions, since all scattering paths and thereby all scattering components are fully determined by $h_i$, $\phi_i$ and $\theta_i$ which are needed to compute the attenuation.
3. Light Scattering from Filaments

Fig. 3.7: The left diagram shows the distribution of scattered light at a perfect smooth dielectric circular cylinder obtained with Monte Carlo particle tracing. The intensity of the scattered light is computed with respect to $\xi$ and $\Delta s$ at the cylindrical shell: the darker the grey, the higher the outgoing flux through the surface. A single light source is illuminating a small rectangular surface patch at $s = 0$ from direction $\varphi = \pi$ and $\theta = 0.2$. The direct surface reflection component (R), the forward scattered component (TT) and the first order caustic (TRT) are clearly and sharply visible. The right image shows the same scene but a dielectric cylinder with volumetric scattering. As a consequence both TT and TRT components are blurred. Note that the structure of the scattering distribution (i.e. the position of the peaks and their relative intensities) does not change.

Having $n$ such components the BFSDF may be factorized:

$$f_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \approx \sum_{j=1}^{n} f_{\text{BFSDF}}^{j}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o)$$

$$= \frac{\delta_{\theta}(\theta_i + \theta_o)}{\cos^2 \theta_i} \sum_{j=1}^{n} \delta_{\chi}^{j}(\Delta s - \lambda_{\chi}^{j}(h_o, \varphi_o, \theta_o)) \delta_{\varphi}^{j}(\varphi_i - \lambda_{\varphi}^{j}(h_o, \varphi_o, \theta_o)) \delta_{h}^{j}(h_i - \lambda_{h}^{j}(h_o, \varphi_o, \theta_o)) a^{j}(h_o, \varphi_o, \theta_o).$$ (3.14)

The first factor $\delta_{\theta}$ says that intensity can only be measured, if $\theta_o = -\theta_i$. Since no blurring of the scattered ray of light occurs, all other factors describing the scattering geometry can be formalized by Dirac delta distributions, too. Because rays
entering the enclosing cylinder at $s_i$ propagate into longitudinal direction, they usually exit the enclosing cylinder at another longitudinal position $s$. The functions $\lambda_s^j$ account for this relationship between $s_i$ and $s$ and the two functions $\lambda_{\phi}^j$ and $\lambda_{\psi}^j$ characterize the actual scattering geometry (the paths of the projection of the ray paths onto the normal plane). The factor $a^j$ is the attenuation for the $j$-th component. These attenuation factors include Fresnel factors as well as absorption. Due to Bravais Law (App. 7.2) all $\lambda^j$ and $a^j$ can be directly derived from the 2D analysis of the optics at the cross section of a fiber within the normal plane together with a modified index of refraction (depending on the incoming inclination $\theta$, see next section).

Note that the factor $\cos^2 \theta$ in front of (3.14) compensates for:

- the integration measure with respect to $\theta, h_i$ and $\phi_i$ which contributes a factor of $\cos^{-1} \theta$ and

- the cosine of the angle of incidence which gives another $\cos^{-1} \theta$.

Even though this kind of BFSDF is for smooth dielectric materials with absorption and locally constant cross section only, its basic structure is very instructive for a lot of other types of fibers, too. For example internal volumetric scattering or surface scattering—due to surface roughness—do not change this basic structure in most cases but result in spatial and angular blurring, cf. Fig. 3.7. By replacing the $\delta$-distributions in the BFSDF for smooth dielectric cylinders—cf.(3.15)—with normalized lobes (like Gaussians) centered over the peak similar effects can be simulated easily.

The presence of sharp peaks in the scattering distribution is not only a characteristic of dielectric fibers with constant cross section, but of most other types of (even not dielectric) fibers with locally varying cross section shape, too. If the variation is periodic and of a very high frequency these peaks are typically shifted or blurred in longitudinal directions, compared to a fiber without this high frequency detail. Nevertheless Bravais Law only holds for constant cross sections. Hence (3.14) has to be adopted to other cases. At least it can be seen as a basis for efficient compression schemes and a starting point for analytical BFSDFs.
Fig. 3.8: **Left:** Azimuthal scattering geometry of the first three scattering modes (R, TT, TRT) within the normal plane for a fiber with circular cross section. The relative azimuths are denoted $\phi_R, \phi_{TT}$ and $\phi_{TRT}$. Here the projection of the angle of incidence onto the normal plane is $\gamma'_i$ and the angle of refraction $\gamma'_r$ respectively. Note that the outgoing ray leaves the fiber always at $-h_i$. **Right:** Longitudinal scattering geometry of the first three scattering modes (R, TT, TRT) at a dielectric fiber. Note that the outgoing inclinations $\theta_R, \theta_{TT}$ and $\theta_{TRT}$ always equal $-\theta_i$. The difference of the longitudinal outgoing positions with respect to incoming position $s_i$ are denoted $s_R, s_{TT}$ and $s_{TRT}$.

### 3.6.1 Dielectric Cylinder

In section 2.1.2 it was shown that all important features of light scattering from human hair fibers can be basically explained by a glass cylinder model. Moreover, this model is an adequate approximation for many other kinds of fibers. Hence, it is very instructive to analyze the scattering from a cylindric fiber made of colored dielectric material, which was already partially discussed in [3], [52] and [37].

Consider a solid dielectric cylindric fiber of radius $r$. We restrict ourselves to the first three scattering components (Fig. 4.22), dominating the visual appearance and denote them according to section 2.1.2:

- The first backward scattering component, a direct (white) surface reflection (R-component)
3.6. Dielectric Fibers

- The first forward scattering component, light that is two times transmitted through the cylinder (TT-component)

- The second backward scattering component, light which enters the cylinder and gets internally reflected (TRT-component)

Hence, according to (3.14), the resulting BFSDF is a superposition of three independent scattering functions, each accounting for one of the three modes:

\[
\int_{\text{BFSDF}} f_{\text{cylinder}} \approx \frac{\delta \theta}{\cos^2 \theta_i} (\delta_{\theta} \delta_{\phi} \delta_{h} a_{\text{R}} + \delta_{\phi} \delta_{TT} \delta_{h} a_{TT} + \delta_{\phi} \delta_{TRT} \delta_{h} a_{TRT}).
\]

(3.15)

Neglecting the attenuation factors \(a_{\text{R}}, a_{\text{TT}},\) and \(a_{\text{TRT}}\) this BFSDF can be derived by tracing the projections of rays within the normal plane (Fig. 4.22 Left). Rays entering the cylinder at an offset \(h_i\) always exit at \(-h_i\). Furthermore intensity can only be measured, if the relative azimuth \(\phi = M[\phi_o - \phi_i]\) equals \(\phi_{\text{R}}\) for the R-component, \(\phi_{\text{TT}}\) for the TT-component or \(\phi_{\text{TRT}}\) for the TRT-component. Here, the operator \(M\) maps the difference angle \(\phi_o - \phi_i\) to an interval \((-\pi, \pi]\):

\[
M[\alpha] = \begin{cases} 
\alpha, & -\pi < \alpha \leq \pi \\
\alpha - 2\pi, & \pi < \alpha \leq 2\pi \\
\alpha + 2\pi, & -2\pi \leq \alpha \leq -\pi 
\end{cases}
\]

(3.16)

For the relative azimuths the following equations hold, cf. Fig. 4.22 Left:

\[
\phi_{\text{R}} = 2\gamma_i' \quad (3.17)
\]

\[
\phi_{\text{TT}} = M[\pi + 2\gamma_i' - 2\gamma_t'] \quad (3.18)
\]

\[
\phi_{\text{TRT}} = 2\gamma_i' - 4\gamma_t' \quad (3.19)
\]

Due to Bravais Law all azimuths are calculated from 2D-scattering within the normal plane with help of a modified index of refraction

\[
n' = \frac{\sqrt{n^2 - \sin^2 \theta_i}}{\cos \theta_i}
\]
(instead of the relative refractive index $n$) and Snell’s Law (App. 7.2, 7.1.2). The projected (signed) angle of incidence light is denoted $\gamma_i'$ and equals $\arcsin h_i$. Its corresponding (signed) angle of refraction is $\gamma'_t = \arcsin(h_i/n')$.

Because rays entering the cylinder at a position $s_i$ propagate into longitudinal $s$-direction, they leave the fiber at another positions $s_{TT}$ and $s_{TRT}$, respectively. In case of direct surface reflection (R-mode) the light is reflected at the incoming position $s_i$. The two positions $s_{TT}$ and $s_{TRT}$ are directly related to the length covered by the ray inside the cylinder. For the TT component this length $l$ can be calculated from its projection onto the normal plane $l_s$, and the longitudinal angle of refraction $\theta_t$ (see Fig. 4.22):

$$l = \frac{l_s \cos \theta_t}{\cos \theta_i} = \frac{2r \cos \gamma_i'}{\cos \theta_i}$$  \hspace{1cm} (3.20)

with $\theta_i = -\sgn(\theta_i) \arccos((n'/n) \cos \theta_i)$ (see also App. 7.2). This length doubles for rays of the TRT-mode.

Therefore for the positions $s_{TT}$ and $s_{TRT}$ the following holds:

$$s_{TT} = -\sgn(\theta_i) \sqrt{l^2 - l_s^2} = l_s \tan \theta_t$$  \hspace{1cm} (3.21)

$$s_{TRT} = 2l_s \tan \theta_t$$  \hspace{1cm} (3.22)

Putting all geometric analysis together the geometry terms $\lambda$ introduced in the previous section are:

$$\lambda_s^R = 0; \lambda_s^{TT} = s_{TT}; \lambda_s^{TRT} = s_{TRT}$$

$$\lambda_h^R = \lambda_h^{TT} = \lambda_h^{TRT} = -h_o$$

$$\lambda_{\phi}^R = \phi_o + 2 \arcsin h_o$$

$$\lambda_{\phi}^{TT} = \phi_o + M \left[ \pi + 2 \arcsin(h_o) - 2 \arcsin(h_o/n') \right]$$

$$\lambda_{\phi}^{TRT} = \phi_o - 4 \arcsin(h_o/n') + 2 \arcsin h_o$$  \hspace{1cm} (3.23)

For smooth dielectric cylinders the R component gets attenuated by Fresnel reflectance (see App 7.1.3) only:

$$a^R = \text{Fresnel}_R$$
According to [37] the index $n'$ is used to calculate the reflectance for perpendicular polarized light, whereas another index of refraction $\tilde{n}' = n^2/n'$ is used for parallel polarized light.

For the two other modes (TT, TRT) a ray gets attenuated at each reflection rsp. refraction event. Hence for the corresponding Fresnel factors $\text{Fresnel}_{TT}$ and $\text{Fresnel}_{TRT}$ the following equations hold:

\[
\text{Fresnel}_{TT} = (1 - a^R)(1 - F(1/n', 1/\tilde{n}', \gamma_t'))
\]
\[
\text{Fresnel}_{TRT} = \text{Fresnel}_{TT} F(1/n', 1/\tilde{n}'', \gamma_t')
\]

If a ray enters the cylinder, as is the case for both the TT and TRT modes, absorption takes place. For homogenous materials the attenuation only depends on the length of the internal path—i.e. the absorption length—and the absorption coefficient $\sigma$. For the TT component the absorption length equals $l$ and a ray of the TRT mode covers twice this distance inside the cylinder. Note that the absorption length given in [37] (which was $l = (2 + 2\cos\gamma_t')/\cos\theta_t$) is wrong and should be replaced by (3.20).

Hence the total attenuation factors with absorption are

\[
a^{TT} = \text{Fresnel}_{TT} e^{-\sigma l},
\]
\[
a^{TRT} = \text{Fresnel}_{TRT} e^{-2\sigma l}.
\]

Note that higher order scattering can be analyzed in a straight forward way, since analytical solutions for the corresponding geometry terms $\lambda$ are available.

3.7 Far-Field Approximation and BCSDF

“Real world” filaments are typically very thin and long structures. Hence, compared to its effective diameter both viewing and lighting distances are typically
very large. Therefore all surface patches along the normal plane have equal $(\phi_o, \theta_o)$ and the fibers cross section is locally illuminated by parallel light (of constant radiance) from a fixed direction $(\phi_i, \theta_i)$. Furthermore adjacent surface patches of the local minimum enclosing cylinder have nearly the same distance to both light source and observer.

When rendering such a scene the screen space width of a filament is less or in the order of magnitude of a single pixel. In this case fibers can be well approximated by curves having a cross section proportional to the diameter of the minimum enclosing cylinder $D$. That allows us to introduce a simpler scattering formalism to further reduce rendering complexity which we call far-field approximation.

Using similar quantities as Marschner et al. [37]—curve radiance and curve irradiance—we show how an appropriate scattering function can be derived right from the BFSDF. The curve radiance $d\bar{L}_o$ is the averaged outgoing radiance along the width of the fibers minimum enclosing cylinder times its effective diameter (Fig. 3.6 Right).
3.7. Far-Field Approximation and BCSDF

Assuming parallel light from direction \((\varphi_i, \theta_i)\) being constant across the width of a fiber this curve radiance can be computed by averaging the outgoing radiance across the width:

\[
\bar{dL}_o(\varphi_i, \theta_i, \varphi_o, \theta_o) := \frac{D}{\pi} \int_{\varphi_o - \pi/2}^{\varphi_o + \pi/2} \cos(\xi_o - \varphi_o) \ dL_o(\varphi_i, \theta_i, \xi_o, \alpha_o(\theta_o, h_o(\xi_o, \varphi_o)), \beta_o(\theta_o, h_o(\xi_o, \varphi_o))) d\xi_o.
\]

(3.29)

Here a factor of \(\cos(\xi_o - \varphi_o)\) accounts for the projected width of the minimum enclosing cylinder. Notice that for the integration with respect to \(\xi_o\) one has to take into account its transition of from zero to \(2\pi\) properly. Using the other set of variables this can be transformed to:

\[
\bar{dL}_o(\varphi_i, \theta_i, \varphi_o, \theta_o) = \frac{D}{2} \int_{-1}^{1} dL_o(h_o, \varphi_i, \theta_i, \varphi_o, \theta_o) dh_o.
\]

(3.30)

Since we assume constant incident radiance \(L_i(\varphi_i, \theta_i)\), \(dL_o\) may be substituted by:

\[
dL_o(\varphi_i, \theta_i, h_o, \varphi_o, \theta_o) = \cos^2 \theta_i L_i(\varphi_i, \theta_i) d\varphi_i d\theta_i dh_o \int_{-1}^{1} \int_{-\infty}^{\infty} f_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \ d\Delta s \ dh_i
\]

(3.31)

which yields:

\[
\bar{dL}_o(\varphi_i, \theta_i, \varphi_o, \theta_o) = \frac{D}{2} L_i(\varphi_i, \theta_i) \cos^2 \theta_i d\varphi_i d\theta_i \int_{-1}^{1} \int_{-\infty}^{\infty} f_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \ d\Delta s \ dh_i \ dh_y
\]

(3.32)
This scattered radiance $d\bar{L}_o$ is proportional to the incoming curve irradiance $d\bar{E}_i$:

$$d\bar{E}_i(\varphi_i, \theta_i) := DL_i(\varphi_i, \theta_i) \cos^2 \theta_i d\varphi_i d\theta_i.$$  \hfill (3.33)

Thus, a new bidirectional far-field scattering distribution function for curves $f_{BCSDF}$ that assumes distant observer and distant light sources can be defined. It relates the incoming curve irradiance to the averaged outgoing radiance across the width (curve radiance). We will call this far-field approximation the Bidirectional Curve Scattering Distribution Function (BCSDF).

$$f_{BCSDF}(\varphi_i, \theta_i, \varphi_o, \theta_o) := \frac{d\bar{L}_o(\varphi_i, \theta_i, \varphi_o, \theta_o)}{d\bar{E}_i(\varphi_i, \theta_i)}. \hfill (3.34)$$

With (3.32) and (3.33) we obtain:

$$f_{BCSDF}(\varphi_i, \theta_i, \varphi_o, \theta_o) = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \int_{-\infty}^{\infty} f_{BFSDF}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) d\Delta s dh_i dh_o. \hfill (3.35)$$

Finally, in order to compute the total outgoing curve radiance one has to integrate all incoming light over a sphere with respect to $\varphi_i$ and $\theta_i$:

$$\bar{L}_o(\varphi_o, \theta_o) = D \int_{0}^{2\pi} \int_{-\pi/2}^{\pi/2} f_{BCSDF}(\varphi_i, \theta_i, \varphi_o, \theta_o) L_i(\varphi_i, \theta_i) \cos^2 \theta_i d\theta_i d\varphi_i. \hfill (3.36)$$

This equation matches the rendering integral given in [37, equation 1] in the specific context of hair rendering. Thus the BCSDF is identical to the fiber scattering function very briefly introduced in [37]. In contrast to [37] our derivation shows the close connection between the BFSDF and the BCSDF. In our formalism the
BCSDF is just one specific approximation of the BFSDF and can be computed directly from (3.55).

Besides of the advantage of being less complex than the BFSDF the BCSDF can help to drastically reduce sampling artifacts which would be introduced by subtle BFSDF detail like very narrow scattering lobes. Nevertheless there are some drawbacks. First of all local scattering effects are neglected which restricts the possible fields of application of the BCSDF in case of close-ups (Fig. 3.9). Furthermore, since the fiber properties are averaged over the width, the entire width—at least in the statistical average—has to be visible. Otherwise undesired artifacts may occur. Finally the BCSDF is not adequate for computing indirect illumination due to multiple fiber scattering in dense fiber clusters, since the far-field assumption is violated in such a case.

3.8 Further Special Cases

Although a distant observer and distant light sources are commonly assumed for rendering, there may be cases where only one of these two assumptions is fulfilled. For these cases further approximations of the BFSDF are straight forward and can be derived similarly to the BCSDF. In particular we now discuss the following two special cases:

- a close observer and locally constant incident lighting: near field scattering with constant incident lighting
- a distant observer with locally varying incident lighting: curve scattering with locally varying incident lighting

Other special cases arise in the context of low frequency lighting. Suppose for example an incident illumination given by a spherical harmonics representation [54]. In this case it is useful to pre-compute scattering with respect to the spherical harmonics basis functions. Based on such pre-computed scattering functions the exitant radiance can be estimated very quickly according to the linear combination of incident lighting.
3.8.1 Near-Field Scattering with Constant Incident Lighting

A BCSDSDF may be a good approximation for distant observers, but it fails when it comes to close-ups. Since the outgoing radiance is averaged across the width of a filament local scattering details cannot be resolved. However, if incident illumination can be assumed locally constant (like in the case of distant light sources) the outgoing radiance \( dL(\phi_i,\theta_i,h_o,\phi_o,\theta_o) \) can be computed from (3.31). Integration with respect to incident direction \((\phi_i,\theta_i)\) yields:

\[
L_o(h_o,\phi_o,\theta_o) = \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} f_{\text{nearfield}} L_i \cos^2 \theta_i d\theta_i d\phi_i.
\]  

(3.37)

with

\[
f_{\text{nearfield}}(\phi_i,\theta_i,h_o,\phi_o,\theta_o) = \int_{-1}^{1} \int_{-\infty}^{\infty} f_{\text{BFSDF}}(\Delta s,h_i,\phi_i,\theta_i,h_o,\phi_o,\theta_o) d\Delta s dh_i.
\]  

(3.38)

A practical example of a near-field scattering function for dielectric fibers is given in section 3.10.2.

3.8.2 Curve Scattering with Locally Varying Incident Lighting

Although locally constant incident lighting may be assumed in most cases of direct illumination, this situation may change in case of indirect illumination. For the sake of completeness we give the result for curve scattering with varying lighting, too. For the rendering integral one obtains:

\[
\bar{L}_o(\phi_o,\theta_o) = D \int_{-\infty}^{\infty} \int_{-1}^{1} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} f_{\text{varyingCSDF}} L_i \cos^2 \theta_i d\theta_i d\phi_i dh_i d\Delta s
\]  

(3.39)
with

\[
f_{\text{varyingCSDF}}(\Delta s, h_i, \varphi_i, \theta_i, \varphi_o, \theta_o) = \frac{1}{2} \int_{-1}^{1} f_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \, dh_o.
\]

(3.40)

### 3.9 Previous Fiber Scattering Models

Especially in the context of hair rendering several models for light scattering from fibers have been proposed. We now show, how they can be expressed in our notation which for instance allows systematic derivations of further scattering functions basing on the corresponding BFSDF and BCSDF. Furthermore we discuss their physical plausibility.

#### 3.9.1 Kajiya & Kay’s Model

One of the first simple approaches to render hair, which is still very commonly used, was presented by Kajiya & Kay [25]. It assumes a distant observer, since the outgoing radiance is constant over the width of a fiber and basically accounts for two scattering components: a scaled specular ad-hoc Phong reflection at the surface of the fiber, centered over the specular cone and an additional colored diffuse component. The diffuse coefficient \( K_{\text{diffuse}} \) is obtained from averaging the outgoing radiance of a diffuse BRDF over the illuminated width of the fiber, which produces significantly different results compared to the actual solution derived in section 3.10.1 (cf. Fig. 3.17). Moreover, the phenomenological Phong reflection fails to predict the correct intensities due to Fresnel reflectance (cf. section 3.10.1).

Kajiya & Kay’s model can be represented by a BCSDF as follows:

\[
f_{\text{Kajiya&Kay}} = K_{\text{Phong}} \frac{\cos^n \left( \frac{\theta_o + \theta_i}{2} \right)}{\cos \theta_i} + K_{\text{diffuse}}.
\]

(3.41)
3. Light Scattering from Filaments

Fig. 3.10: Intensity measurements of [37]. **Left:** Keeping the light source fixed ($\theta_i = 45^\circ$) and measuring the scattering lobes along the tangent for different relative azimuths $\phi$. Note the presence of the three highlights: the primary R-highlight, the secondary TRT-highlight and the TT-highlight. **Right:** Sweeping around the specular cone while keeping the light source fixed. Due to the elliptical cross section of the fiber the glints (strong intensity peaks) depend on the orientation of the cross section.

3.9.2 The Model of Marschner et al. (2003)

A much more sophisticated far-field model for light scattering from hair fibers was proposed by Marschner et al. [37]. It bases on significant measurements (see also Fig. 3.10) of light scattering from single hair filaments and implies again a distant observer and distant light sources. In [37] it is shown that all important features of light scattering from hairs can be basically explained by scattering from cylindrical dielectric fibers made of colored glass accounting for the first three scattering components (R, TT, TRT) already introduced in section 3.6. However, the derivation of the model presented by Marschner et al. is rather awkward and omits interesting details.

In the following we will take advantage of our unified scattering formalism. The BCSDF representing the basic model of Marschner et al. (without smoothing the caustic in the TRT component) can be directly derived from its underlying
BFSDF, which is given by:

\[
\begin{align*}
\frac{\delta(h_o + h_i)}{\cos^2 \theta_d} & \left( g(\theta_o + \theta_i - \Delta \theta_R, w^\theta_R) d^R(\theta_d, h_i) \delta(\Delta s) \delta(\phi_i - \lambda^R) 
\right. \\
+ & \left. g(\theta_o + \theta_i - \Delta \theta_{TT}, w^\theta_{TT}) a^TT(\theta_d, h_i) \delta(\Delta s - \lambda^{TT}) \delta(\phi_i - \lambda_{TT}) 
\right) \\
+ & \left. g(\theta_o + \theta_i - \Delta \theta_{TRT}, w^\theta_{TRT}) a^{TRT}(\theta_d, h_i) \delta(\Delta s - \lambda^{TRT}) \delta(\phi_i - \lambda_{TRT}) \right). 
\end{align*}
\]

This BFSDF can be seen as a generator for the basic model proposed in [37]. Note that it matches the BFSDF for a cylindric dielectric fiber with normalized Gaussians \( g(x, w) := 1/(\sqrt{2\pi}w) \exp(-x/(2w^2)) \) replacing the delta distributions \( \delta_\theta \). This accounts for the fact that no fiber is perfectly smooth, and the light is scattered to a finite lobe around the perfect specular cone. Additional shifts \( \Delta \theta_{R, TT, TRT} \) account for the tiled surface structure of hair fibers, which lead to shifted specular cones compared to a perfect dielectric fiber. For a better phenomenological match in [37] it is furthermore proposed to replace \( \theta_i \) with \( \theta_d = (\theta_o - \theta_i)/2 \). Hence this BFSDF is just a variation of (3.15) and the complex derivation of the basic model for a smooth fiber given in [37] can be reduced to the following general recipe:

- Build a BFSDF by writing the scattering geometry in terms of products of delta functions with geometry terms \( \lambda_\phi \) and adding an additional attenuation factor.

- Derive the corresponding BCSDF according to (3.55). All ray density factors given in [37] as well as multiple scattering paths for the TRT-component are a direct consequence of the \( \lambda_\phi \)-terms within the \( \delta \)-functions.

Note that for performing the integration of \( \frac{\delta(h_o + h_i)}{\cos^2 \theta_d} \) in (3.55) symbolically, the following rule has to be applied:
with $G \in \{ \mathbb{R}, \text{TT}, \text{TRT} \}$. Here the expression $f^G(h_o)$ subsumes all factors depending on $h_o$ (except the $\delta$-function itself) and $h_o^i$ denotes the i-th root of the expression $\varphi_i - \lambda^{G}_\varphi(h_o)$. For both the R and the TT component there exists always exactly one root. For the TRT component one needs a case differentiation, since it exhibits either one or three roots. This exactly matches the observation of [37] that one or three different paths for rays of the TRT component occur. Since the function $\phi(h_o) := \varphi_i - \lambda^{\text{TRT}}_\varphi(h_o)$ is smooth, the transition from one to three roots represents a fold in $\phi(h_o)$. This fold occurs when $\frac{d\phi}{dh_o} = \frac{d\lambda^{\text{TRT}}_\varphi(h_o)}{dh_o} = 0$, hence if $h^{1,2}_o = \pm \sqrt{4 - n'^2/3}$. However, this causes symmetric singularities of $I^\text{TRT}_h$ which reflects the fact that at $h^{1,2}_o$ the ray density goes to infinity. This pair of caustics is responsible for the glints in human hair, which were already phenomenologically introduced in section 2.1.2.

Although it is basically possible to compute the roots $h^i_o$ for all components directly, it is—for the sake of efficiency—useful to apply the approximation given by [37, equation 10].

As an example we now have a look at the direct surface reflection component (R-component). According to (3.42) one has:

$$f_R^{\text{BFSDF}} = \frac{a^R(\theta_d, h_i)}{\cos^2 \theta_d} \delta(h_o + h_i) g(\theta_o + \theta_i - \Delta \theta_R, w_R^\theta) \delta(\Delta s) \delta(\varphi_i - \lambda^{R}_\varphi)$$

Inserting this BFSDF into (3.55) for the corresponding BCSDF yields:

$$f_R^{\text{BCSDF}} = \frac{a^R(\theta_d, \frac{\phi}{2}) \cos(\frac{\phi}{2}) g(\theta_o + \theta_i - \Delta \theta_R, w_R^\theta)}{4 \cos^2 \theta_d}. \quad (3.44)$$

This equation matches the formula given in [37, equation 8] (for $p = 0$, with $\phi =$
$M[\varphi_o - \varphi_i]$ and inserting the ray density and attenuation factors). A more detailed derivation of all scattering components is given in the appendix (App. 7.4).

In [37] the basic BCSDF is extended to account for elliptic fibers and to smooth singularities in a post processing step. Since the BCSDF is computed for a perfect smooth cylinder, there are azimuthal angles $\phi_c$, for which the intensities of the TRT-components go to infinity. These singularities (caustics) are unrealistic and have to be smoothed out for real world fibers. The method used in [37] is to first remove the caustics from the scattering distribution and to replace them by Gaussians representing roughly the same portion of energy and centered over the caustics positions. Four different parameters are needed to control this caustic removal process.

Note that this rather awkward caustic handling for the TRT component performed in [37] as well as a problematic singularity in the ray density factor of the TT-component for an index of refraction approaching the limit of one can be avoided, if another BFSDF for non-smooth dielectric fibers is used, cf. section 3.10.2.

Usually hair has not a circular but has an elliptic cross section geometry. Since even mild eccentricities $e$ especially influence the azimuthal appearance of the TRT component, this fact must not be neglected. In [37] the following simple approximation is proposed: Substitute an index $n^*$ for the original refractive index $n$ depending on $\varphi_h = (\varphi_i + \varphi_o)/2$ for the TRT-component as follows:

$$n^*(\varphi_h) = \frac{1}{2} ((n_1 + n_2) + \cos(2\varphi_h)(n_1 - n_2))$$

where

$$n_1 = 2(n - 1)e^2 - n + 2$$
$$n_2 = 2(n - 1)e^{-2} - n + 2.$$  

3.10 Examples of Further Analytic Solutions and Approximations

In general it cannot be expected that for complex BFSDF/BCSDF there are analytical solutions—a situation that is similar to the one for BSSRDF/BSDF. However, in the following we derive analytical solutions or at least analytical approximations for some interesting special cases.
3. Light Scattering from Filaments

3.10.1 Opaque Circular Symmetric Fibers

Perfectly opaque fibers with a circular cross section like wires can be modeled by a
generalized cylinder together with a BRDF characterizing its surface reflectance.
For this common case we now show, how both BFSDF and BCSDF can be de-
erived. Since no internal light transport takes place, light is only reflected from a
surface patch, if it is directly illuminated. Thus the BFSDF simply writes as a
product of the BRDF of the surface and two additional delta distributions limiting
the reflectance to a single surface patch:

\[
f_{\text{BFSDF}} = f_{\text{BRDF}} \delta(\xi_o - \xi_i) \delta(\Delta s) \quad (3.45)
\]
or with respect to the other set of variables:

\[
f_{\text{BFSDF}} = f_{\text{BRDF}} \delta(\phi + \arcsin h_o - \arcsin h_i) \delta(\Delta s) \quad (3.46)
\]

with \( \phi = M[\varphi_o - \varphi_i] \) being the relative azimuth. To derive the BCSDF the term
\( \delta(\phi + \arcsin h_o - \arcsin h_i) \) has to be transformed into a form suitable for inte-
gration with respect to \( h_i \) first. Applying the properties of Dirac distributions one
obtains

\[
f_{\text{BFSDF}} = f_{\text{BRDF}} |\cos(\phi + \arcsin h_o)| \delta(h_i - \sin(\phi + \arcsin h_o)) \delta(\Delta s). \quad (3.47)
\]

According to (3.55) for the BCSDF of a circular fiber mapped with an arbitrary
BRDF the following holds:

\[
f_{\text{BCSDF}} = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \int_{-\infty}^{\infty} f_{\text{BRDF}} |\cos(\phi + \arcsin h_o)| \delta(h_i - \sin(\phi + \arcsin h_o)) \delta(\Delta s)
\]
\[\times d\Delta s d h_i d h_o \]
\[= \frac{1}{2} \int_{-1}^{1} f_{\text{BRDF}} |\sin(\phi + \arcsin h_o)| \cos(\phi + \arcsin h_o) dh_o \quad (3.48)
\]
Some exemplary results are shown in figure 3.17.

**Lambertian Reflectance**

Many analytical BRDFs include a diffuse term accounting for Lambertian reflectance. The BCSDF approximation for such a Lambertian BRDF ($f_{\text{BRDF}}^{\text{Lambertian}} = k_d$) component can be directly calculated from (3.48):

$$f_{\text{BCSDF}}^{\text{Lambertian}} = \frac{k_d}{2} \int_{-1}^{\cos \phi} | \cos (\phi + \arcsin h_o) | dh_o$$  \hspace{1cm} (3.49)

with $k_d$ denoting the diffuse reflectance coefficient. To solve this integral we replace the integrand by the absolute value of its Taylor series expansion about $h_o = 0$ up to an order of two:

$$f_{\text{BCSDF}}^{\text{Lambertian}} \approx \frac{k_d}{2} \int_{-1}^{\cos \phi} | \cos \phi - h_o \sin \phi - \frac{h_o^2 \cos \phi}{2} | dh_o. \hspace{1cm} (3.50)$$

For the resulting approximative BCSDF, which has a relative error of less than five percent compared to the original, the following equations holds:

$$f_{\text{BCSDF}}^{\text{Lambertian}} \approx \frac{k_d}{2} \left| \frac{5}{6} \cos \phi + \cos^2 \phi - \frac{1}{6} \cos^4 \phi + \frac{1}{2} \sin \phi - \frac{1}{2} \sin \phi \cos^2 \phi \right|. \hspace{1cm} (3.51)$$

**Fresnel Reflectance**

A second very important feature of opaque fibers like metal wires or coated plastics is Fresnel reflectance. We analyzed such a surface reflectance in the context of scattering from dielectric fibers in section 3.6.1 (R-component) and the corresponding BCSDF was already derived in section 3.9.2, cf. (7.36). In fact the BRDF
is averaged over the width of a fiber, the BCSDF of a fiber with a narrow normalized reflection lobe BRDF (instead of a dirac delta distribution) can be very well approximated by this BCSDF for smooth fibers. Some exemplary scenes rendered with a combination of a Lambertian and a Fresnel BRDF can be seen in Fig. 3.17. The BCSDF approximation produces very similar results compared to the precise BRDF (BFSDF) solution, but required roughly 16 times less rendering time.

3.10.2 A Practical Parametric Near-Field Shading Model for Dielectric Fibers with Elliptical Cross Section

In the following we derive a flexible and efficient near-field shading model for dielectric fibers according to section 3.8.1. This model accurately reproduces the scattering pattern for close ups but can be computed much more efficiently than particle tracing (or an equivalent) that was typically used to capture the scattering pattern correctly.

As a basis, we take the three component BFSDF given by (3.42) that was also used to derive the BCSDF corresponding to the basic model proposed in [37], cf. section 3.9.2. Due to surface roughness and inhomogeneities inside the fiber the scattering distribution gets blurred (spatial and angular blurring), cf. Fig. 3.7. In order to account for this effect we replace all $\delta$-distributions with special normalized Gaussians $g(I, x, w) := N(I, w) \exp(-x^2/(2w^2))$ with a normalization factor of $N(I, w) := 1 / \int_{-I}^{I} \exp(-x^2/(2w^2)) \, dx$. The widths $w$ of these Gaussians control the “strength of bluriness”. Furthermore, we add a diffuse component approximating higher order scattering.

With these modifications we obtain the following more general BFSDF for non-smooth dielectric fibers:

$$
\begin{align*}
\int_{\text{BFSDF}}^\text{dielectric} &= g(1, h_o + h_i, w_h) \\
&= \frac{g(1, h_o + h_i, w_h)}{\cos^2 \theta_d} \\
&\times \left( g\left(\frac{\pi}{2}, \theta_o + \theta_i - \Delta \theta_R, w_R^\theta \right) d^R(\theta_d, h_i) \\
&+ g\left(\pi, \theta_o + \theta_i - \Delta \theta_{TT}, w_{TT}^\theta \right) d^{TT}(\theta_d, h_i) \\
&+ g\left(\pi, \theta_o + \theta_i - \Delta \theta_R, w_R^\lambda \right) d^R(\theta_d, h_i) \\
&+ g\left(\pi, \theta_o + \theta_i - \Delta \theta_{TT}, w_{TT}^\lambda \right) d^{TT}(\theta_d, h_i) \\
&+ g\left(\pi, \theta_o + \theta_i - \Delta \theta_R, w_R^\phi \right) d^R(\theta_d, h_i) \\
&+ g\left(\pi, \theta_o + \theta_i - \Delta \theta_{TT}, w_{TT}^\phi \right) d^{TT}(\theta_d, h_i) \right)
\end{align*}
$$
Note that the awkward caustic handling done in [37] can be avoided by calculating the corresponding BCSDF \( f_{\text{dielectric BCSDF}} \) out of \( f_{\text{dielectric BFSDF}} \), cf. section 3.9.2. In this case the appearance of the caustics is fully determined by the azimuthal blurring width \( w_{\text{TRT}} \). The more narrow this width is, the closer the caustic pattern matches the one of a smooth fiber, thus the stronger the glints. All azimuthal scattering of such a BCSDF can be derived efficiently by numerical integration in a preprocessing pass. The result is a two dimensional lookup table with respect to \( \theta_d \) and \( \phi \) which is then used for rendering and which replaces the complex computations proposed in 3.9.2. Some exemplary comparisons between BFSDF and BCSDF-renderings are presented in Fig. 3.16.

Assuming locally constant illumination a near-field scattering function \( f_{\text{dielectric nearfield}} \) can be derived from the BFSDF by solving eqn. 3.38 in section 3.8.1 for \( f_{\text{dielectric BFSDF}} \). Assuming a narrow width \( w_h \), i.e. a small spatial blurring, this approximately yields:

\[
f_{\text{dielectric nearfield}} = 2k_d + \frac{1}{\cos^2 \theta_d} \left( g\left( \frac{\pi}{2}, \theta_o + \theta_i - \Delta \theta_R, w_{\theta R}^\phi \right) a_R^{\theta R} (\theta_d, h_o) \right. \\
+ g\left( \frac{\pi}{2}, \theta_o + \theta_i - \Delta \theta TT, w_{\theta TT}^\phi \right) a TT (\theta_d, h_o) \\
\left. + g\left( \pi, \phi + 2 \arcsin h_o, w_{\theta R}^\phi \right) \\
+ g\left( \pi + M \left[ \pi + 2 \arcsin h_o - 2 \arcsin \left( h_o/n' \right) \right], w_{\theta TT}^\phi \right) \right)
\]

(3.53)

This result can be directly used for shading of fibers with a circular cross sec-
3. Light Scattering from Filaments

Fig. 3.11: Absorption for two different “hair colors” (blond and black) according to http://www.solarlaser.com/doc/photoepilation\_1\_en.htm.

Although elliptic fibers could be modeled by calculating the corresponding \(\lambda_{\varphi}\) expressions, we propose to use the following more efficient approximation instead: Change the diameter of the primitive used for rendering—a half cylinder or a flat polygon—according to the projected diameter and apply the first order approximation for mild eccentricities proposed in [37] (cf. section 3.9.2).

Results obtained with this model are given in the results section (Fig. 3.13, 3.14, 3.15, 3.18, 3.19). Notice especially the results shown in Fig. 3.13, in which comparisons to photographs are given showing how well the scattering patterns of the renderings are matching the originals.

Interestingly, this near-field shading model is computationally less expensive than the far-field model of [37]. It can be evaluated more than twice as fast (per call) compared to an efficient reimplementation of this far-field model. Furthermore an extension of the model to all higher order scattering components is possible, since analytical expressions for corresponding geometry terms \(\lambda_{\varphi}\) are available.
Tab. 3.1: Summary of all parameters of the near-field shading model with plausible values for hair (see also [37], [57], Fig. 3.11)

<table>
<thead>
<tr>
<th>notation</th>
<th>meaning</th>
<th>typical plausible values for human hair</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \theta_R$</td>
<td>longitudinal shifts of the R-lobe</td>
<td>-10° to -5°</td>
</tr>
<tr>
<td>$\Delta \theta_{TT}$</td>
<td>longitudinal shifts of the TT-lobe</td>
<td>$-\Delta \theta_R/2$</td>
</tr>
<tr>
<td>$\Delta \theta_{TRT}$</td>
<td>longitudinal shifts of the TRT-lobe</td>
<td>$-3\Delta \theta_R/2$</td>
</tr>
<tr>
<td>$w^\theta_R$</td>
<td>widths for longitudinal blurring of the R-lobe</td>
<td>5° to 10°</td>
</tr>
<tr>
<td>$w^\theta_{TT}$</td>
<td>widths for longitudinal blurring of the TT-lobe</td>
<td>$w^\theta_R$</td>
</tr>
<tr>
<td>$w^\theta_{TRT}$</td>
<td>widths for longitudinal blurring of the TRT-lobe</td>
<td>0.5° to 3°</td>
</tr>
<tr>
<td>$w^\psi_R$</td>
<td>widths for azimuthal blurring of the R-lobe</td>
<td>$2w^\theta_R$</td>
</tr>
<tr>
<td>$w^\psi_{TT}$</td>
<td>widths for azimuthal blurring of the TT-lobe</td>
<td>$w^\psi_R$</td>
</tr>
<tr>
<td>$w^\psi_{TRT}$</td>
<td>widths for azimuthal blurring of the TRT-lobe</td>
<td>$w^\psi_R$</td>
</tr>
<tr>
<td>$n$</td>
<td>index of refraction</td>
<td>1.55</td>
</tr>
<tr>
<td>$r$</td>
<td>strand radius</td>
<td>0.0025 – 0.006 cm</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>index of absorption in $cm^{-1}$</td>
<td>&gt; 5</td>
</tr>
<tr>
<td>$k_d$</td>
<td>diffuse coefficient</td>
<td>$0.01 \cdot \exp(-3r\sigma)$</td>
</tr>
<tr>
<td>$e$</td>
<td>eccentricity</td>
<td>0.85 to 1</td>
</tr>
</tbody>
</table>

### 3.11 Extending the BFSDF

Although the BFSDF is a very accurate approach to describe scattering from filaments of in principle any cross section, it may be advantageous to decouple information on the cross section geometry from scattering, especially if the cross sections’ convex hull strongly deviates from the perimeter (Fig. 3.12 Left). In such cases it is more likely that an overlap of minimum enclosing cylinders would introduce a substantial bias. Also the conventional parameterization might not be the best choice for analytical BFSDF models.

However, this issue can be solved by slightly modifying the original BFSDF. The idea is to parameterize all scattering with respect to the projected width of the
Fig. 3.12: Extending the BFSDF. Left: Conventional parameterization of a BFSDF at the perimeter. Right: A modified parameterization based on the projected width \(2r(\varphi)\) of the fibers cross section. The offset \(h\) is normalized according to \(r(\varphi)\).

actual cross section instead of the perimeter. For a given azimuthal angle \(\varphi\) this width is computed as the normal projection of the cross section with respect to \(\varphi\) (Fig. 3.12 Right).

Let \(r(\varphi)\) denote half the projected width and assume that the offset \(h\) is now normalized accordingly. Then a modified scattering function \(f'_{\text{BFSDF}}\) may be for example defined by the following rendering integral:

\[
L_o(s_o, h_o, \varphi_o, \theta_o) = \frac{1}{r(\varphi_o)} \int_{-\infty}^{+\infty} \int_{-1}^{1} \int_{0}^{\pi/2} \int_{-\pi/2}^{\pi/2} r(\varphi) f'_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \cos^2 \theta_i L_i(s_i, h_i, \varphi_i, \theta_i) \, d\theta_i \, d\varphi_i \, dh_i \, d\Delta s.
\]

Based on this definition it is possible to derive similar far-field approximations as for the BFSDF. Assuming directional incident lighting and a distant observer one obtains the following BCSDF analogon \(f'_{\text{BCSDF}}\):

\[
f'_{\text{BCSDF}}(\varphi_i, \theta_i, \varphi_o, \theta_o) = \frac{r(\varphi_i)}{2r(\varphi_o)} \int_{-\infty}^{+\infty} \int_{-1}^{1} \int_{-\infty}^{+\infty} f'_{\text{BFSDF}}(\Delta s, h_i, \varphi_i, \theta_i, h_o, \varphi_o, \theta_o) \, d\Delta s \, dh_i \, dh_o.
\]
Note that for rendering one has to account for the projected widths which may depend on $\varphi_i$ and $\varphi_o$. For a conventional BFSDF/BCSDF it was enough to know the diameter of the minimum enclosing cylinder.

### 3.12 Conclusion

In this chapter we derived a novel theoretical framework for efficiently computing light scattering from filaments. In contrast to previous approaches developed in the realm of hair rendering it is much more flexible and can handle many types of filaments and fibers like hair, fur, ropes and wires. Approximations for different levels of abstraction can be derived in a straight forward way. Furthermore, our approach provides a firm basis for comparisons and classifications of filaments with respect to their scattering properties. The basic idea was to adopt basic radiometric concepts like BSSRDF and BSDF to the realm of fiber rendering. Although the resulting radiance transfer functions are strictly speaking valid for infinite fibers only, they can be seen as suitable local approximations.

However, the theoretical analysis given in this thesis should be seen as a starting point and further work is required to investigate practical issues which arise in the case of “real world filaments”. Moreover, efficient compression strategies are needed to reduce the memory costs for storing Bidirectional Fiber Scattering Distribution Functions. Also, if the BFSDF is given in a purely numerical form, the four dimensional rendering integral itself is very costly to evaluate. Therefore, the BFSDF should be approximated by series of analytical functions which can be efficiently integrated. Fortunately, many scattering distributions have very few intensity peaks which can be well approximated with existing compression schemes (e.g. Lafortune Lobes [11], Reflectance Field Polynomials [42], Linear Basis Decomposition [38], clustering and factorization [26, 33, 49]) developed in the realm of BRDF and BTF rendering.

In this chapter several aspects of light scattering from a single fiber were analyzed. However, also inter-reflections between fibers (multiple fiber scattering)
play an important role for the overall color. In the next chapter this issue will be discussed in detail. It will be shown that the BFSDF is the key to efficient physically based global illumination of fiber assemblies.

3.13 Results

*Fig. 3.13:* An oval made of rough plastic illuminated by one point light source. The upper row shows photographs and the second row renderings with a variant of the model presented in section 3.10.2. The parameters are set to roughly match the original photographs. Note the similarity of both positions and intensities of the highlights for different viewing angles. It took less than five seconds to render each frame with conventional ray tracing.
3.13. Results

Fig. 3.14: Various BFSDF examples with a more complex illumination.

Fig. 3.15: Two different close-ups of hair rendered with a mental ray® implementation (cf. http://www.mentalimages.com) of the near-field model presented in section 3.10.2.
Fig. 3.16: Different strands of translucent dielectric fibers illuminated by three distant point light sources. The left two leftmost images were rendered with a hair like material and the two rightmost images show synthetic fibers. Direct illumination was computed according to section 3.10.2. The BCSDF was computed from the BFSDF for dielectric fibers (cf. section 3.10.2). Please note how well the BCSDF approximations match the original BFSDF renderings.
Fig. 3.17: **Top row:** Various physically based (i.e. energy conserving) BCSDF renderings of opaque fibers illuminated by three distant point light sources. The BCSDFs were generated according to the following BRDF types: Lambertian (*left*); Fresnel + (1 − Fresnel) × Lambertian (*left middle*); Fresnel for aluminium (*right middle*); Fresnel for gold (*right*). These BCSDF examples are given in a purely analytical form, so that parameters—like index of refraction—can be easily changed on the fly, e.g. for the “gold” and “aluminium” examples, which use the same BCSDF with different material properties. **Middle row:** BFSDF renderings of the same scenes. Notice how well the BCSDF renderings match the BFSDF originals. **Bottom row:** Some exemplary renderings with the phenomenological model of Kajiya & Kay. Lighting and geometry was not changed. The diffuse component was computed according to the diffuse coefficient of the BRDF. The width of the specular lobe (if present) as well as the peak intensity was set to match the ones of the original physically based renderings setting the Fresnel factor to one. Nevertheless, the results differ substantially from the BCSDF resp. BFSDF renderings. Note that the BCSDF images were computed approximately one order of magnitude faster, since 16× less eye-rays per pixel were needed to achieve a similar rendering quality.
Fig. 3.18: BFSDF renderings (based on the model presented in section 3.10.2) of a hair style consisting of about 90000 individual hair fibers (left: brown; middle: blond; right: dark red dyed) illuminated by five point light sources. The images were rendered in about 15 hours each with Monte Carlo path tracing (see section 4.7).
Fig. 3.19: BFSDF rendering of a fabric made of dielectric fibers using the model presented in section 3.10.2. It took eight hours to render the image with Monte Carlo path tracing (see section 4.7).
3. Light Scattering from Filaments