# Gelfand pairs

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# Introduction

This research is devoted to a very interesting and important class of homogeneous spaces of real Lie groups. We suppose that all considered homogeneous spaces are Riemannian manifolds. This assumption allows us to use a lot of geometrical methods.

Let $X = G/K$ be a connected Riemannian homogeneous space of a real Lie group $G$. We assume that the action $G : X$ of $G$ on $X$ is locally effective, i.e., $K$ contains no non-trivial connected normal subgroups of $G$. Denote by $\mathcal{D}(X)^G$ the algebra of $G$-invariant differential operators on $X$ and by $\mathcal{P}(T^*X)^G$ the algebra of $G$-invariant functions on $T^*X$ polynomial on fibres. It is well known that $\mathcal{P}(T^*X)^G$ is a Poisson algebra, the Poisson bracket being induced by the commutator in $\mathcal{D}(X)^G$.

**Definition 1.** The homogeneous space $X$ is called *commutative* or the pair $(G, K)$ is called a *Gelfand pair* if the following five equivalent conditions are satisfied:

(0) the algebra $\mathcal{D}(X)^G$ is commutative;

(1) the algebra of $K$-invariant measures on $X$ with compact support is commutative with respect to convolution;

(2) the algebra $\mathcal{P}(T^*X)^G$ is commutative with respect to the Poisson bracket;

(3) the representation of $G$ on $L^2(X)$ has a simple spectrum;

(4) the action of $G$ on $T^*X$ is coisotropic with respect to the standard symplectic structure on the cotangent vector bundle.

Condition (1) was first considered by Gelfand in [18]. The equivalence of (0) and (1) is proved by Thomas [42] and Helgason [19], independently. Clearly, (0) implies (2). The inverse implication is proved by Rybnikov [40]. The equivalence of (1) and (3) is proved e.g. by Berezin et al. in [5]. Finally, the equivalence of (2) and (4) is proved by Vinberg [43]. Good references for the theory of Gelfand pairs are [16] and [43].

Symmetric Riemannian homogeneous spaces introduced by Élie Cartan are commutative. In case $X$ is compact, this was proved by Cartan himself. The theory of symmetric spaces is well developed. Works of Élie Cartan [12] and Sigurdur Helgason [19], [20], [21] describe their structure and also deal with harmonic analysis on such manifolds. The common eigenfunctions of $\mathcal{D}(X)^G$ that are invariant under $K$ are called spherical functions on $X$. Many special functions arise in this way.

There is a more general geometrical condition sufficient for commutativity. In his celebrated work [41] on the trace formula, Selberg introduced a notion of weakly symmetric homogeneous space. He proved that each weakly symmetric homogeneous space is commutative.

Let $\sigma$ be an automorphism of $G$ such that $\sigma(K) = K$. Define an automorphism $s$ of $X$ by $s(gK) := \sigma(g)K$.

**Definition 2.** The homogeneous space $X$ is said to be *weakly symmetric with respect to* $\sigma$, if for every pair of points $x, y \in X$ there is $g \in G$ such that $gx = sy$, $gy = sx$. $X$ is said to be *weakly symmetric*, if it is weakly symmetric with respect to some automotphism $\sigma$.

Selberg pointed out that his "trace formula" is true not only for the weakly symmetric spaces but for all commutative ones as well. He did not know if the second class of spaces is strictly larger. In 2000, Lauret [26] constructed the first example of a commutative but not weakly symmetric homogeneous space.

Due to lack of non-trivial examples weakly symmetric homogeneous spaces were forgotten for almost 30 years. Clearly, each symmetric space is weakly symmetric. It is well known that the second class is larger. For example, as was noticed by Selberg, a homogeneous space $(\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SO}_2)/\mathrm{SO}_2$ is weakly symmetric but not symmetric. Recently weakly symmetric homogeneous spaces were intensively studied by several mathematicians, see, for example, [1], [6], [7], [8], [9]. In particular, new examples of non-symmetric weakly symmetric homogeneous spaces have been constructed. These works show that weakly symmetric spaces possess a fairly interesting geometry.

**Definition 3.** A real or complex linear Lie group with finitely many connected components is said to be *reductive* if it is completely reducible.

Let $F$ be a complex reductive Lie group and $H \subset F$ a reductive subgroup.

**Definition 4.** An affine complex $F$-variety $X$ is called *spherical* if a Borel subgroup $B(F) \subset F$ has an open orbit in $X$. If $X$ is a linear space and a spherical $F$-variety then it is called a spherical representation of $F$. If a homogeneous space $F/H$ is spherical, then the pair $(F, H)$ and the subgroup $H$ are also called spherical.

Let $G$ be a real form of a complex reductive group $G(\mathbb{C})$. Suppose $K \subset G$ is a compact subgroup. We call the real homogeneous space $G/K$, the subgroup $K$ and the pair $(G, K)$ spherical if the complexification $X(\mathbb{C}) = G(\mathbb{C})/K(\mathbb{C})$ is a spherical $G(\mathbb{C})$-variety. In case of reductive $G$ the notions of commutative and weakly symmetric homogeneous spaces are equivalent, see [1]. Moreover, weakly symmetric spaces are real forms of complex affine spherical homogeneous spaces, [1]. These latter spaces are classified by Krämer [25] (if $G(\mathbb{C})$ is simple), Brion [10] and Mikityuk [30] (if $G(\mathbb{C})$ is semisimple).

Homogeneous space $X$ of a reductive Lie group $G$ is spherical if and only if each $G$-invariant Hamiltonian system on $T^*X$ is integrable within the class of Noether integrals, see [30]. In case of non-reductive $G$, the notion of spherical homogeneous space does not exist. The notions of weakly symmetric and commutative homogeneous spaces are natural substitutes for it. It would be interesting to realise whether this result of [30] extends to all Gelfand pairs.

Note that the classifications in [10, 30] are not quite complete, because the case of non-principal homogeneous spaces is not treated there. This gap is fixed here, see Theorem 2.1. The real forms of homogeneous spherical spaces, i.e., commutative homogeneous spaces of *real reductive* groups are explicitly described in Section 2.1. We obtain many new examples of weakly symmetric Riemannian manifolds. Most of them are not symmetric regardless of the (or under some particular) choice of a $G$-invariant Riemannian metric.

The principal result of this work is the complete classification of Gelfand pairs. Our main tool in obtaining classification is a criterion for commutativity of homogeneous spaces, which is also useful and interesting in its own right.

Fix some notation that will be used throughout the text. Lie algebras of Lie groups are denoted by corresponding small Gothic letters; for instance, $\mathfrak{n} = \mathrm{Lie}\, N$. Unless otherwise explicitly stated, all Lie groups, algebras, vector spaces are assumed to be real. If $G$ is a Lie group, then $G^0$ is the identity component of $G$; $G'$ is the commutator group of $G$; $Z(G)$ is the connected centre of $G$;

$G(\mathbb{C})$ is the complexification of a real Lie group $G$;

$\mathbb{R}[X]$ is the algebra of real-valued regular function on an affine algebraic variety $X$;

$\mathbb{R}[X]^G$ is the subalgebra of $G$-invariants in $\mathbb{R}[X]$.

If a reductive Lie group $F$ acts on a linear space $V$, then $F_*(V)$ denotes a generic stabiliser for this action and $F_y$ the stabiliser of $y \in V$.

Some necessary conditions for the commutativity of arbitrary homogeneous spaces are due to Vinberg. If $G/K$ is commutative, then, up to a local isomorphism, $G$ has a factorisation $G = N \rtimes L$, where $N$ is the nilpotent radical of $G$, $K \subset L$, $L$ and $K$ have the same invariants in $\mathbb{R}[\mathfrak{n}]$, and $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = 0$, see [43]. Without loss of generality, one may assume that $L'$ is a real form of a complex semisimple group and the centre of $L$ is compact. Hence, $L$ is a reductive group.

The following is our commutativity criterion.

**Theorem 1.** *$X = (N \rtimes L)/K$ is commutative if and only if all of the following three conditions hold:*

*(A) $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$;*

*(B) for any point $\gamma \in \mathfrak{n}^*$ the homogeneous space $L_\gamma/K_\gamma$ is commutative;*

*(C) for any point $\beta \in (\mathfrak{l}/\mathfrak{k})^*$ the homogeneous space $(N \rightthreetimes K_\beta)/K_\beta$ is commutative.*

It is always assumed below that $G = N \rightthreetimes L$, $K \subset L$, and $X = G/K$ is a commutative homogeneous space. Let $P$ denote the *ineffective kernel* of the action $L : \mathfrak{n}$. Then $P$ is a normal subgroup of $L$ and $G$. Let us indicate some important consequences of Theorem 1. By condition (A), we have $L/P \subset \mathrm{O}(\mathfrak{n})$. Hence, $L_\gamma$ is reductive for any $\gamma \in \mathfrak{n}^*$. Therefore condition (B) actually means that $L_\gamma/K_\gamma$ is spherical, and one can use classification results for spherical homogeneous spaces. Because the orbits of the compact group $K$ in $\mathfrak{n}$ are separated by polynomial invariants, $L$ and $K$ have the same invariants in $\mathbb{R}[\mathfrak{n}]$ if and only if they have the same orbits. In other words, condition (A) means that there is a factorisation $L = L_*(\mathfrak{n})K$ or, equivalently, $L/P$ is a product of $L_*(\mathfrak{n})/P$ and $K/(K \cap P)$. All non-trivial factorisations of compact groups into products of two subgroups are classified by Onishchik [32] (see also [34, Chapter 4]).

Our classification is based on the following principles and conventions.

- A homogeneous space $G/K$ is called *indecomposable* if it cannot be presented as a product $G_1/K_1 \times G_2/K_2$, where $G = G_1 \times G_2$, $K = K_1 \times K_2$ and $K_i \subset G_i$. Obviously, $G_1/K_1 \times G_2/K_2$ is commutative if and only if both spaces $G_1/K_1$ and $G_2/K_2$ are commutative. Hence, it suffices to classify only indecomposable commutative homogeneous spaces.
- Commutativity is a local property, i.e., it depends only on the pair of algebras $(\mathfrak{g}, \mathfrak{k})$, see [43]. Therefore we may assume that $G$ and $K$ are connected, $N$ is simply connected, $L = Z(L) \times L_1 \times \ldots \times L_m$, where $Z(L)$ is the connected centre of $L$, and the $\{L_i\}$'s are the simple factors of $L$. We also assume that $L_i$ are real forms of simply connected complex simple groups and the action of $Z(L)/(Z(L) \cap P^0)$ on $\mathfrak{n}$ is effective. Given a pair $(\mathfrak{g}, \mathfrak{k})$, it may happen that there is no effective pair $(G, K)$ satisfying these assumptions, so we admit not only effective actions $G : (G/K)$, but locally effective as well.
- Assume that $\mathfrak{z}_0 \subset [\mathfrak{n}, \mathfrak{n}]$ is an $L$-invariant subspace, and $Z_0 \subset N$ is the corresponding connected subgroup. Then the homogeneous space $X/Z_0 = ((N/Z_0) \rightthreetimes L)/K$ is also commutative, see [43]. The passage from $X$ to $X/Z_0$ is called a *central reduction.* A commutative homogeneous space is said to be *maximal*, if it can not be obtained by a non-trivial central reduction from a larger one. Clearly, one can consider only maximal commutative homogeneous spaces.
- In Chapters 1 and 3 we impose on $G/K$ two technical conditions. The first of them concerns the behaviour of $Z(K)$ with respect to the simple factors of $L$ and the action $Z(L) : \mathfrak{n}$. Let $\mathfrak{n}/\mathfrak{n}' = \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_p$ be a decomposition of the $L$-module $\mathfrak{n}/\mathfrak{n}'$ into irreducibles. Since $P$ is a normal subgroup of $L$, it is reductive. We say that $G/K$ is *principal*, if $P$ is semisimple, $Z(K) = Z(L) \times (L_1 \cap Z(K)) \times \ldots \times (L_m \cap Z(K))$, and $Z(L) = C_1 \times \ldots \times C_p$, where $C_i \subset \mathrm{GL}(\mathfrak{w}_i)$. The second condition, "$\mathrm{Sp}_1$-saturation", describes the behaviour of normal subgroups of $K$ and $L$ isomorphic to $\mathrm{Sp}_1$. The precise definition is given in Section 1.5. Both these constrains are removed in Chapter 4.

In Chapter 1, we obtain a partial classification of Gelfand pairs. The simplest and most important results are obtained for simple $L$. Note that if $L$ is simple, then $G/K$ is automatically principal and $\mathrm{Sp}_1$-saturated.

Denote by $H_n$ the $(2n+1)$-dimensional real Heisenberg group. In tables and theorems we write $U_n$ instead of $U_1 \times \mathrm{SU}_n$ and sometimes $\mathrm{SO}_n$ instead of $\mathrm{Spin}_n$. The symbols $\mathbb{R}^n$ and $\mathbb{C}^n$ stand for simply-connected Abelian groups, which are regarded as standard $L$-modules.

**Theorem 2.** *Suppose $X = (N \leftthreetimes L)/K$ is an indecomposable commutative space, where $L$ is simple, $L \neq K$ and $\mathfrak{n} \neq 0$. Then $X$ is one of the following eight spaces.*

$$\begin{array}{lll} (H_{2n} \leftthreetimes \mathrm{SU}_{2n})/\mathrm{Sp}_n; & (\mathbb{R}^7 \leftthreetimes \mathrm{SO}_7)/G_2; & ((\mathbb{R}^8 \times \mathbb{R}^2) \leftthreetimes \mathrm{SO}_8)/\mathrm{Spin}_7; \\ (\mathbb{C}^{2n} \leftthreetimes \mathrm{SU}_{2n})/\mathrm{Sp}_n; & (\mathbb{R}^8 \leftthreetimes \mathrm{Spin}_7)/\mathrm{Spin}_6; & (\mathbb{R}^8 \leftthreetimes \mathrm{SO}_8)/\mathrm{Spin}_7; \\ (\mathbb{R}^{2n} \leftthreetimes \mathrm{SO}_{2n})/\mathrm{SU}_n; & & (\mathbb{R}^8 \leftthreetimes \mathrm{SO}_8)/(\mathrm{Sp}_2 \times \mathrm{SU}_2). \end{array}$$

Commutativity of $(N \leftthreetimes L)/K$ implies that of $(N \leftthreetimes (L/P))/(K/(K \cap P))$. Therefore we first describe the indecomposable commutative spaces with trivial $P$ (Sections 1.2 and 1.3) and then study possible kernels $P$ in Section 1.4. The classification results of Sections 1.2–1.4 can be summarised as follows.

**Theorem 3.** *Let $X = (N \leftthreetimes L)/K$ be a maximal principal indecomposable commutative homogeneous space and $L_1 \lhd L$ a simple direct factor acting on $\mathfrak{n}$ non-trivially. If $L_1 \neq \mathrm{SU}_2$ and $L_1 \not\subset K$, then either $L$ is simple (and $X$ appears in Theorem 2) or $X$ is one of the following four spaces:*

$(H_{2n} \leftthreetimes \mathrm{U}_{2n})/(\mathrm{Sp}_n \cdot \mathrm{U}_1)$, $\quad (H_8 \leftthreetimes (\mathrm{SO}_8 \times \mathrm{U}_1))/(\mathrm{Spin}_7 \times \mathrm{U}_1)$;

$((\mathbb{R}^n \leftthreetimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$, $\quad ((H_n \leftthreetimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$.

*(In the last two items, the normal subgroups $\mathrm{SO}_n$ and $\mathrm{SU}_n$ of $K$ are diagonally embedded into $\mathrm{SO}_n \times \mathrm{SO}_n$ and $\mathrm{SU}_n \times \mathrm{SU}_n$, respectively.)*

A commutative homogeneous space $(N \leftthreetimes L)/K$ is said to be of *Heisenberg type* if $L = K$. The following is the main classification result of Chapter 1.

**Theorem 4.** *Any indecomposable maximal principal $\mathrm{Sp}_1$-saturated commutative homogeneous space belongs to the one of the three classes:*
*1) the commutative homogeneous spaces of reductive real Lie groups;*
*2) the homogeneous spaces listed in Theorems 2 and 3.*
*3) the commutative homogeneous spaces of Heisenberg type.*

Chapter 2 is devoted to commutative spaces of reductive Lie groups. As we have already mentioned, they are real forms of spherical affine homogeneous spaces. Let $(G(\mathbb{C}), H)$ be a spherical pair of complex reductive groups. We describe real commutative spaces corresponding to $G(\mathbb{C})/H$. Assume that $G(\mathbb{C})$, $H$, and $G$ are connected. The subgroup $K$ is a maximal compact subgroup of $H$. For $G$ we can take any real form of $G(\mathbb{C})$ containing $K$. The subgroup $K$ is always contained in a maximal compact subgroup of $G(\mathbb{C})$. For a non-compact $G$, we have the following result.

**Theorem 5.** *$G = (G(\mathbb{C})^{\tau\varphi})^0$, $K = H^\tau$, where $\varphi$ is an involution of $G(\mathbb{C})$ acting trivially on $H$, and $\tau$ is a compact real structure on $G(\mathbb{C})$, commuting with $\varphi$ and preserving $H$.*

Recall that in case of reductive $G$ the notions of commutative and weakly symmetric spaces are equivalent.

Denote by $\mathrm{Aut}(G, K)$ the set of automorphisms of $G$ preserving $K$. We call an automorphism $\sigma \in \mathrm{Aut}(G, K)$ *righteous*, if it defines a weakly symmetric structure on $X = G/K$, i.e., if $X$ is weakly symmetric with respect to $\sigma$.

Suppose weakly symmetric homogeneous space $X = G/K$ is a real form of a spherical affine homogeneous space $Y = G(\mathbb{C})/H$. Denote by $V_\lambda$ the irreducible representation of $G(\mathbb{C})$ with the highest weight $\lambda$. Let

$$\mathbb{C}[Y] = \bigoplus_{\lambda \in \Lambda(Y)} V_\lambda,$$

be the decomposition into irreducible representations of $G(\mathbb{C})$. Note that this decomposition is canonical [45].

The following theorem characterise all righteous automorphisms of weakly symmetric homogeneous spaces of reductive Lie groups.

**Theorem 6.** *An automorphism $\sigma \in \mathrm{Aut}(G, K)$ is righteous if and only if $\sigma(V_\lambda) = V_\lambda^*$ for each weight $\lambda \in \Lambda(Y)$.*

Let $X = G/K$ be weakly symmetric with respect to $\sigma$. One can introduce a $G$-invariant Riemannian metric on $X$. This metric will be also $\sigma$-invariant. The Riemannian manifold $X$ can be symmetric even if $X$ is not symmetric as a homogeneous space of $G$. For example, an odd dimensional sphere $\mathrm{S}^{2n-1} = \mathrm{SU}_n/\mathrm{SU}_{n-1} = \mathrm{SO}_{2n}/\mathrm{SO}_{2n-1}$ is a symmetric Riemannian manifold and simultaneously a non-symmetric weakly symmetric homogeneous space of $\mathrm{SU}_n$. To understand whether a given Riemannian metric is symmetric, it is sufficient to know the isometry group of the pair $(X, \mu)$ or its identity component $P = \mathrm{Isom}(X)^0$. Let $Q$ be the stabiliser of $eK \in X$ in $P$. Clearly, there is a factorisation $P = GQ$. The Riemannian manifold $X$ is symmetric if and only if $Q$ is a symmetric subgroup of $P$. Factorisations of reductive groups into products of two reductive subgroups are described by Onishchik [32], [33]. This allows us to classify non-symmetric weakly symmetric Riemannian manifolds with reductive isometry group. In case of a non-compact $X$ we prove the following theorem.

**Theorem 7.** *An indecomposable (as a homogeneous space) non-symmetric non-compact homogeneous space of a semisimple group $G$ is not a symmetric Riemannian manifold regardless of the choice of a $G$-invariant metric.*

Classification results in the compact case are presented in Table 2.6 (Subsection 2.2.4).

Chapter 3 is devoted to homogeneous spaces of Heisenberg type. Recently, these spaces were intensively studied by several people, see [3], [27], [31], [43], [44]. We complete the classification of principal $\mathrm{Sp}_1$-saturated commutative spaces of Heisenberg type, started in [3] and [43], [44].

Since $L = K$, we have $\mathcal{D}(G/K)^G \cong U(\mathfrak{n})^K$, where $U(\mathfrak{n})$ is the universal enveloping algebra of $\mathfrak{n}$. In particular, if $\mathfrak{n}$ is an Abelian Lie algebra, then $G/K$ is commutative. It is called a commutative space of *Euclidian type*. Such a space is completely determined by a $K$-module structure on $\mathfrak{n}$. Therefore, there is no harm in assuming that $\mathfrak{n}$ is not Abelian. Recall that $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = 0$ if $G/K$ is commutative. For the spaces of Heisenberg type, it was already proved in [3]. Commutative homogeneous spaces of Heisenberg type such that $\mathfrak{n}/\mathfrak{n}'$ is a simple $K$-module are classified in [43] and [44]. In general, $\mathfrak{n}$ is a sum of an Abelian ideal and algebras listed in [43, Table 3] and [44]. But the problem of classifying possible sums is not trivial.

Interest of commutative spaces of Heisenberg type is explained by their connections with spherical representations. Recall relevant structure results. Consider a homogeneous space $(N \rightthreetimes K)/K$, where $\mathfrak{n}$ is two-step nilpotent and $\dim[\mathfrak{n},\mathfrak{n}] = 1$. Set $\mathfrak{z} = [\mathfrak{n},\mathfrak{n}]$. Decompose $\mathfrak{n}$ into a $K$-invariant direct sum $\mathfrak{n} = (\mathfrak{w}\oplus\mathfrak{z})\oplus V$, where $V$ is an Abelian ideal and $\mathfrak{w}\oplus\mathfrak{z}$ is a Heisenberg algebra. A nonzero covector $\alpha \in \mathfrak{z}^*$ determines a non-degenerate skew-symmetric form $\hat\alpha$ on $\mathfrak{w}$; namely, $\hat\alpha(\xi,\eta) = \alpha([\xi,\eta])$ for $\xi,\eta \in \mathfrak{w}$. Therefore the complexification of $\mathfrak{w}$, $\mathfrak{w}(\mathbb{C})$, is simultaneously an orthogonal and symplectic $K(\mathbb{C})$-module. Hence, $\mathfrak{w}(\mathbb{C}) \simeq W\oplus W^*$ for some $K(\mathbb{C})$-module $W$. By [3] and [48], $(N \rightthreetimes K)/K$ is commutative if and only if $W$ is a spherical representation of the complexification of $K_*(V)$. In the simplest situation when $V = 0$ this means that $W$ is a spherical representation of $K(\mathbb{C})$. Classification of spherical representations was obtained by combined efforts of Kac [22], Brion [11], Benson and Ratcliff [4], and Leahy [28] (see historical comments in [24]).

The list of commutative homogeneous spaces $(N \rightthreetimes K)/K$, where $N$ is a product of several Heisenberg groups, is given in [3]. That article also claims to classify all commutative homogeneous spaces $(N \rightthreetimes K)/K$ such that $\mathfrak{n} = \mathfrak{n}_0 \oplus V$, where $V$ is an Abelian ideal and $\mathfrak{n}_0$ is a direct sum of several Heisenberg algebras. The authors of [3] erroneously assume that if $N_0 \subset N$ is the subgroup with Lie $N_0 = \mathfrak{n}_0$ and $(N_0 \rightthreetimes K)/K$ is commutative, then $(N \rightthreetimes K)/K$ is commutative as well. The simplest counterexample is $((\mathbb{C}^2 \times H_2) \rightthreetimes \mathrm{SU}_2)/\mathrm{SU}_2$. By [43, Prop. 15], this space is not commutative, whereas $(H_2 \rightthreetimes \mathrm{SU}_2)/\mathrm{SU}_2$ is commutative.

**Theorem 8.** *All indecomposable* $\mathrm{Sp}_1$*-saturated maximal principal commutative homogeneous spaces* $(N \rightthreetimes K)/K$ *with non-commutative* $\mathfrak{n}$ *and reducible* $\mathfrak{n}/\mathfrak{n}'$ *are given in Table 3.2 in a sense that* $\mathfrak{n}$ *is a* $K$*-invariant subalgebra of* $\mathfrak{n}_{max}$.

In Table 3.2, the Lie algebra $\mathfrak{n}_{\max}$ is described in the following way. Each subspace in parentheses represent a subalgebra $\mathfrak{w}_i \oplus [\mathfrak{w}_i,\mathfrak{w}_i]$, where $\mathfrak{w}_i \subset (\mathfrak{n}/\mathfrak{n}')$ is an irreducible $K$-invariant subspace with $[\mathfrak{w}_i,\mathfrak{w}_i] \neq 0$. The spaces given outside parentheses are Abelian.

Notation $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1{\times}\mathrm{Sp}_{n/2})$ means that this normal subgroup of $K$ can be equal to either of these three groups. Appearance of the symbol $\mathrm{Sp}_{n/2}$ means that $n$ is even.

Table 3.2.

| | $K$ | $\mathfrak{n}_{\max}$ |
|---|---|---|
| 1 | $\mathrm{U}_n$ | $(\mathbb{C}^n \oplus \mathbb{R}) \oplus \mathfrak{su}_n$ |
| 2 | $\mathrm{U}_4$ | $(\mathbb{C}^4 \oplus \Lambda^2\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 3 | $\mathrm{U}_1 \times \mathrm{U}_n$ | $(\mathbb{C}^n \oplus \mathbb{R}) \oplus (\Lambda^2\mathbb{C}^n \oplus \mathbb{R})$ |
| 4 | $\mathrm{SU}_4$ | $(\mathbb{C}^4 \oplus HS^2\mathbb{H}^2 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 5 | $\mathrm{U}_2 \times \mathrm{U}_4$ | $(\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus H\Lambda^2\mathbb{C}^2) \oplus \mathbb{R}^6$ |
| 6 | $\mathrm{SU}_4 \times \mathrm{U}_m$ | $(\mathbb{C}^4 \otimes \mathbb{C}^m \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 7 | $\mathrm{U}_m \times \mathrm{U}_n$ | $(\mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{R}) \oplus (\mathbb{C}^m \oplus \mathbb{R})$ |
| 8 | $\mathrm{U}_1 \times \mathrm{Sp}_n \times \mathrm{U}_1$ | $(\mathbb{H}^n \oplus \mathbb{R}) \oplus (\mathbb{H}^n \oplus \mathbb{R})$ |
| 9 | $\mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{U}_1$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus (\mathbb{H}^n \oplus \mathbb{R})$ |
| 10 | $\mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{Sp}_1$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus (\mathbb{H}^n \oplus \mathbb{H}_0)$ |
| 11 | $\mathrm{Sp}_n \times (\mathrm{Sp}_1, \mathrm{U}_1, \{e\}) \times \mathrm{Sp}_m$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus \mathbb{H}^n \otimes \mathbb{H}^m$ |
| 12 | $\mathrm{Sp}_n \times (\mathrm{Sp}_1, \mathrm{U}_1, \{e\})$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus HS_0^2\mathbb{H}^n$ |
| 13 | $\mathrm{Spin}_7 \times (\mathrm{SO}_2, \{e\})$ | $(\mathbb{R}^8 \oplus \mathbb{R}^7) \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$ |
| 14 | $\mathrm{U}_1 \times \mathrm{Spin}_7$ | $(\mathbb{C}^7 \oplus \mathbb{R}) \oplus \mathbb{R}^8$ |
| 15 | $\mathrm{U}_1 \times \mathrm{Spin}_7$ | $(\mathbb{C}^8 \oplus \mathbb{R}) \oplus \mathbb{R}^7$ |
| 16 | $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{Spin}_8$ | $(\mathbb{C}^8_+ \oplus \mathbb{R}) \oplus (\mathbb{C}^8_- \oplus \mathbb{R})$ |
| 17 | $\mathrm{U}_1 \times \mathrm{Spin}_{10}$ | $(\mathbb{C}^{16} \oplus \mathbb{R}) \oplus \mathbb{R}^{10}$ |
| 18 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2}) \times \mathrm{SU}_2$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathfrak{su}_2$ |
| 19 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2}) \times \mathrm{U}_2$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \oplus \mathbb{R})$ |
| 20 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1{\times}\mathrm{Sp}_{n/2}){\times}\mathrm{SU}_2{\times}$ ${\times}(\mathrm{SU}_m, \mathrm{U}_m, \mathrm{U}_1{\times}\mathrm{Sp}_{m/2})$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^m \oplus \mathbb{R})$ |
| 21 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1{\times}\mathrm{Sp}_{n/2}){\times}\mathrm{SU}_2{\times}\mathrm{U}_4$ | $(\mathbb{C}^n{\otimes}\mathbb{C}^2{\oplus}\mathbb{R}){\oplus}(\mathbb{C}^2{\otimes}\mathbb{C}^4{\oplus}\mathbb{R}){\oplus}\mathbb{R}^6$ |
| 22 | $\mathrm{U}_4 \times \mathrm{U}_2$ | $\mathbb{R}^6 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathfrak{su}_2$ |
| 23 | $\mathrm{U}_4 \times \mathrm{U}_2 \times \mathrm{U}_4$ | $\mathbb{R}^6{\oplus}(\mathbb{C}^4{\otimes}\mathbb{C}^2{\oplus}\mathbb{R}){\oplus}(\mathbb{C}^2{\otimes}\mathbb{C}^4{\oplus}\mathbb{R}){\oplus}\mathbb{R}^6$ |
| 24 | $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{SU}_4$ | $(\mathbb{C}^4 \oplus \mathbb{R}) \oplus (\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 25 | $(\mathrm{U}_1{\times})\mathrm{SU}_4({\times}\mathrm{SO}_2)$ | $(\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6 \otimes \mathbb{R}^2$ |

In Chapter 4, we classify non-$\mathrm{Sp}_1$-saturated and non-principal commutative spaces. This classification is done in terms of certain weighted graphs $\Gamma_q$. To each graph the attache a triple $(F, \check{F}, V)$ such that $F = \mathrm{Sp}_1{\times}\check{F}$, $\mathfrak{f} \subset \mathfrak{so}(V)$ and a Lie algebra $\mathfrak{n}$ generated by $V$. Using these data we construct non-$\mathrm{Sp}_1$-saturated commutative spaces. Let us start with the description of the correspondence between graphs $\Gamma_q$ and triples $(F, \check{F}, V)$.

Let $\Gamma_q$ be a connected rooted graph with vertices $0, 1, \ldots, q$, where $0$ is the root, and maybe one *special* vertex $a_s$. Attach to each vertex $i$ a weight $d(i)$, which is either a positive integer or $\infty$. We say that a vertex $i$ is finite if $d(i) < \infty$, an edge $(i, j)$ is finite if both $i$ and $j$ are finite. Assume that $d(0) = d(a_s) = 1$, each infinite vertex has degree 1, and if $(i,j)$ is infinite edge with $d(j)=\infty$, then $d(i)>1$ and there is at most one infinite vertex $t\neq j$ such that $(i,t)$ is an edge of $\Gamma_q$. To each non-special vertex $i$ we attach a group $H(i)=\mathrm{Sp}_{d(i)}$. To the special vertex $a_s$ we attach a group $H(a_s)$ and a linear representation $H(a_s):\check{V}(a_s)$. Moreover, the pair $(H(a_s),\check{V}(a_s))$ is one of the following: $(\mathrm{U}_1,\{0\})$; $(\mathrm{Sp}_1,\mathbb{R}^3)$; $(\mathrm{Sp}_1{\times}(\mathrm{S})\mathrm{U}_4,\mathbb{C}^2{\otimes}\mathbb{C}^4\oplus\mathbb{R}^6)$; $(\mathrm{Sp}_1{\times}(\mathrm{S})\mathrm{U}_m,\mathbb{C}^2{\otimes}\mathbb{C}^m)$ with $m\geq 3$. Set $W_{i,j}:=\mathbb{H}^{d(i)}\otimes_{\mathbb{H}}\mathbb{H}^{d(j)}$ for each finite edge $(i,j)$. If $(i,j)$ is infinite and $d(j)=\infty$, we set $W_{i,j}:=HS^2_0\mathbb{H}^{d(i)}\cong\mathfrak{su}_{2d(i)}/\mathfrak{sp}_{d(i)}$. Since $\Gamma_q$ is connected, here $d(i)<\infty$. Let $H$ be a product of $H(i)$ over all finite vertices, $\check{H}$ a product of $H(i)$ over all finite vertices except for the root, and $W$ a direct sum of $W_{i,j}$ over all edges of $\Gamma_q$. Set $V:=W\oplus\check{V}(a_s)$. Then $\mathfrak{h}\subset\mathfrak{so}(V)$. Suppose $\Gamma_q$ contains no triple edges, then the normaliser $N_{\mathrm{SO}(V)}(\mathfrak{h})$ is locally isomorphic to $(\mathrm{U}_1)^r{\times}H$. We define $F$ to be a product of $(\mathrm{U}_1)^t{\times}H$, where $(\mathrm{U}_1)^t\subset(\mathrm{U}_1)^r$, and set $\check{F}=(\mathrm{U}_1)^t{\times}\check{H}$. Clearly, $\mathfrak{f}\subset\mathfrak{so}(V)$.

Consider a homogeneous space $X=(V\leftthreetimes F)/(\mathrm{U}_1{\times}\check{F})$, where $F=\mathrm{Sp}_1{\times}\check{F}$, $\mathrm{U}_1\subset\mathrm{Sp}_1$, and $V$ is an Abelian Lie group. According to Theorem 1, $X$ is commutative if and only if $F_*(V)=\mathrm{Sp}_1{\times}\check{F}_*(V)$. On the other hand, $F_*(V)=\mathrm{Sp}_1{\times}\check{F}_*(V)$ if and only if the triple $(F,\check{F},V)$ corresponds to a tree $T_q$ which has no special vertices and satisfies the following two conditions:

(I) if $d(i)>1$, then the vertex $i$ has degree at most 2;

(II) if there is an edge $(i,j)$ with $d(i)>1$, $d(j)>1$, then either $i$ or $j$ has degree 1.

We illustrate the structure of $X$ by the following diagram.

$$\mathrm{U}_1 \text{ —— } \mathrm{Sp}_1 \longrightarrow T_q$$

Here the direct factor $\mathrm{Sp}_1$ of $L$ corresponds to the root of $T_q$.

If $\Gamma_q$ contains a special vertex or a double edge we can define an $F$-invariant Lie algebra structure either on $V$ or on $V\oplus\mathbb{R}$ (see Lemma 4.10), i.e., we attach to $\Gamma_q$ a Lie algebra $\mathfrak{n}=\mathfrak{n}(\Gamma_q)$. In case $\Gamma_q$ is a tree with no special vertices $\mathfrak{n}$ is Abelian. The classification of principal maximal non-$\mathrm{Sp}_1$-saturated commutative spaces, which are not of Heisenberg type, is done in terms of trees $T_q$ satisfying conditions (I), (II) and graphs $\Gamma_q$, described in Lemma 4.9. The result is given in Theorem 4.11.

Let $Fr_s$ be a forest of $s$-trees satisfying conditions (I), (II). Let $X_0=(N_0\leftthreetimes K_0)/K_0$ be an $\mathrm{Sp}_1$-saturated commutative space of Heisenberg type with $\mathfrak{n}_0'\neq 0$. Suppose $K_0=Z(L){\times}L_1{\times}\ldots{\times}L_s{\times}L_{s+1}{\times}\ldots{\times}L_m$, where $L_1=L_2=\ldots=L_s=\mathrm{Sp}_1$ and a triple $(F,\check{F},V)$ is attached to $Fr_s$. We assume that each $L_i$ corresponds to the root of the $i$-th tree of $Fr_s$. Set $K=Z(L){\times}L_{s+1}{\times}\ldots{\times}L_m{\times}(\mathrm{Sp}_1)^s{\times}\check{F}$, $\mathfrak{n}=\mathfrak{n}_1\oplus V$, where for each edge $(j,t)$ of the $i$-th tree $[W_{j,t},W_{j,t}]$ is non-zero only if $j=0$ and $\mathfrak{l}_i\subset\mathfrak{n}_0'$, in that case $[W_{0,t},W_{0,t}]$ can be $\mathfrak{l}_i$. (We do not require that $[W_{0,t},W_{0,t}]=\mathfrak{l}_i$ for all edges $(0,t)$ of the $i$-th tree.) One can show that $X=(N\leftthreetimes K)/K$ is commutative. We say that such $X$ is a space of a *wooden* type. We classify those indecomposable commutative homogeneous spaces of Heisenberg type, which are not of wooden type.

Here we come across yet another difficulty. Suppose $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}_0$, $K = \mathrm{Sp}_n \times \mathrm{Sp}_1$. Then both homogeneous spaces $(N \leftthreetimes \mathrm{Sp}_n)/\mathrm{Sp}_n$ and $(N \leftthreetimes (\mathrm{Sp}_n \times \mathrm{Sp}_1))/(\mathrm{Sp}_n \times \mathrm{Sp}_1)$ are commutative. It follows that any representation $(\mathrm{Sp}_1 \times H) : V$ of a compact group $\mathrm{Sp}_1 \times H$ gives rise to a new commutative space $((V \times N) \leftthreetimes (\mathrm{Sp}_n \times \mathrm{Sp}_1 \times H))/(\mathrm{Sp}_n \times \mathrm{Sp}_1 \times H))$. In Theorem 4.15 and Lemma 4.16, we describe several non-$\mathrm{Sp}_1$-saturated commutative spaces of Heisenberg type in terms of graphs $T_q$ and $\Gamma_q$. To conclude the classification of principal commutative spaces, we need the following construction.

Table 4.1.

| $\boxed{\mathrm{Sp}_1} \cdot \mathrm{Sp}_n \ : \ (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus HS_0^2\mathbb{H}^n$ | $\boxed{\mathrm{Sp}_1} \cdot \mathrm{Sp}_n \times \mathrm{Sp}_m \ : \ (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus \mathbb{H}^n \otimes \mathbb{H}^m$ |
|---|---|
| $\boxed{\mathrm{Sp}_1} \cdot \mathrm{Sp}_n \cdot \underset{\substack{\downarrow \\ T_q}}{\mathrm{Sp}_1} \ : \ (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus \mathbb{H}^n$ | $\boxed{\mathrm{Sp}_1} \cdot \underset{\substack{\downarrow \\ T_q}}{\mathrm{Sp}_1} \ : \ \mathbb{H} \oplus \mathfrak{sp}_1$ |

Take $r$ commutative spaces $\hat{X}_i$ containing in Table 4.1. Suppose $\hat{X}_i = (\hat{N}_i \leftthreetimes \hat{K}_i)/\hat{K}_i$ and $\hat{K}_i = \mathrm{Sp}_1 \times H_i$, where $\mathrm{Sp}_1$ is the direct factor in the box. Take any linear representation $V$ of a compact group $(\mathrm{Sp}_1)^s \times F$. Set $K := H \times H_1 \times \ldots \times H_r \times F$, where $H$ is a subgroup of $(\mathrm{Sp}_1)^r \times (\mathrm{Sp}_1)^s$, $\mathfrak{n} := \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_r \oplus V$, where $V$ is a commutative subspace, and let $X = (N \leftthreetimes K)/K$ be a homogeneous space of $G = N \leftthreetimes K$.

**Theorem 9.** *Suppose $X$ is a principal maximal indecomposable non-$\mathrm{Sp}_1$-saturated space of Heisenberg type. Then either $X$ is listed in Theorem 4.15 or Lemma 4.16, or is obtained by the procedure described above.*

In Section 4.2, we describe possible connected centres of $L$ and $K$. We suppose that $P$ is semisimple. If this is not the case, then $G/K$ is commutative if and only if $(G/Z(P))/K$ is commutative, where $Z(P)$ is the connected centre of $P$.

Let $X = G/K$ be a non-principal maximal indecomposable commutative space. We can enlarge groups $L$ and $K$ and obtain a principal commutative space $\widetilde{X}$, such that $\widetilde{L}' = L'$, $\widetilde{K}' = K'$ and $\widetilde{N} = N$. In general $\widetilde{X}$ is decomposable $\widetilde{X} = X_1 \times \ldots \times X_r$. Each $X_i = (\widetilde{N}_i \leftthreetimes \widetilde{L}_i)/\widetilde{K}_i$ is a central reduction (maybe trivial) of a maximal principal indecomposable commutative space. For each $i$ either $\widetilde{L}_i$ or $\widetilde{K}_i$ has a non-trivial connected centre. Suppose we have such a product $\widetilde{X} = X_1 \times \ldots \times X_r$. Let $C_i$ be the connected centre of $\widetilde{L}_i$ and $Z_i$ of $\widetilde{K}_i$. In order to classify all commutative homogeneous spaces, we have to describe all subgroups $Z(L) \subset C_1 \times \ldots \times C_r$ and $Z(K) \subset Z_1 \times \ldots \times Z_r$ such that $(N \leftthreetimes L)/K$, where $L = Z(L) \times \widetilde{L}'$, $K = Z(K) \times \widetilde{K}'$, is commutative. In case of reductive $G$ it was done in Chapter 2, another particular case is considered in [4] and [28].

Via sequence of reductions, our problem is reduced to the situation where $Z(L) \subset Z(K)L'$ and each $(\widetilde{N}_i \leftthreetimes \widetilde{L}'_i)/\widetilde{K}'_i$ is not commutative, if $X_i$ is not of Heisenberg type. We denote by

$\widetilde{X}_{\mathrm{Heis}}$ the product of all direct factors of $\widetilde{X}$, which are of Heisenberg type and by $\widetilde{X}_{\mathrm{red}}$ of all direct factors, which are commutative spaces of reductive (semisimple) groups. Suppose $\widetilde{X} = \widetilde{X}_1 \times \cdots \times \widetilde{X}_s \times \widetilde{X}_{\mathrm{Heis}} \times \widetilde{X}_{\mathrm{red}}$.

Denote by $Z_{\circledast}$ a connected central subgroup of $\widetilde{L}_{\mathrm{Heis}} = \widetilde{K}_{\mathrm{Heis}}$ such that $(\widetilde{L}_{\mathrm{Heis}})_*(\mathfrak{n})\widetilde{L}'_{\mathrm{Heis}} = Z_{\circledast} \times \widetilde{L}'_{\mathrm{Heis}}$. Let $Z(L)$ be a subgroup of $C_1 \times \cdots \times C_s \times Z(\widetilde{L}_{\mathrm{Heis}})$ and $Z(K)$ a subgroup of $Z_1 \times \cdots \times Z_s(\widetilde{K}_{\mathrm{Heis}}) \times Z(\widetilde{K}_{\mathrm{red}})$. Assume that $Z(K)$ is contained in $L := Z(L) \times \widetilde{L}'$ and set $K := Z(K) \times \widetilde{K}'$, $X = (N \rtimes L)/K$.

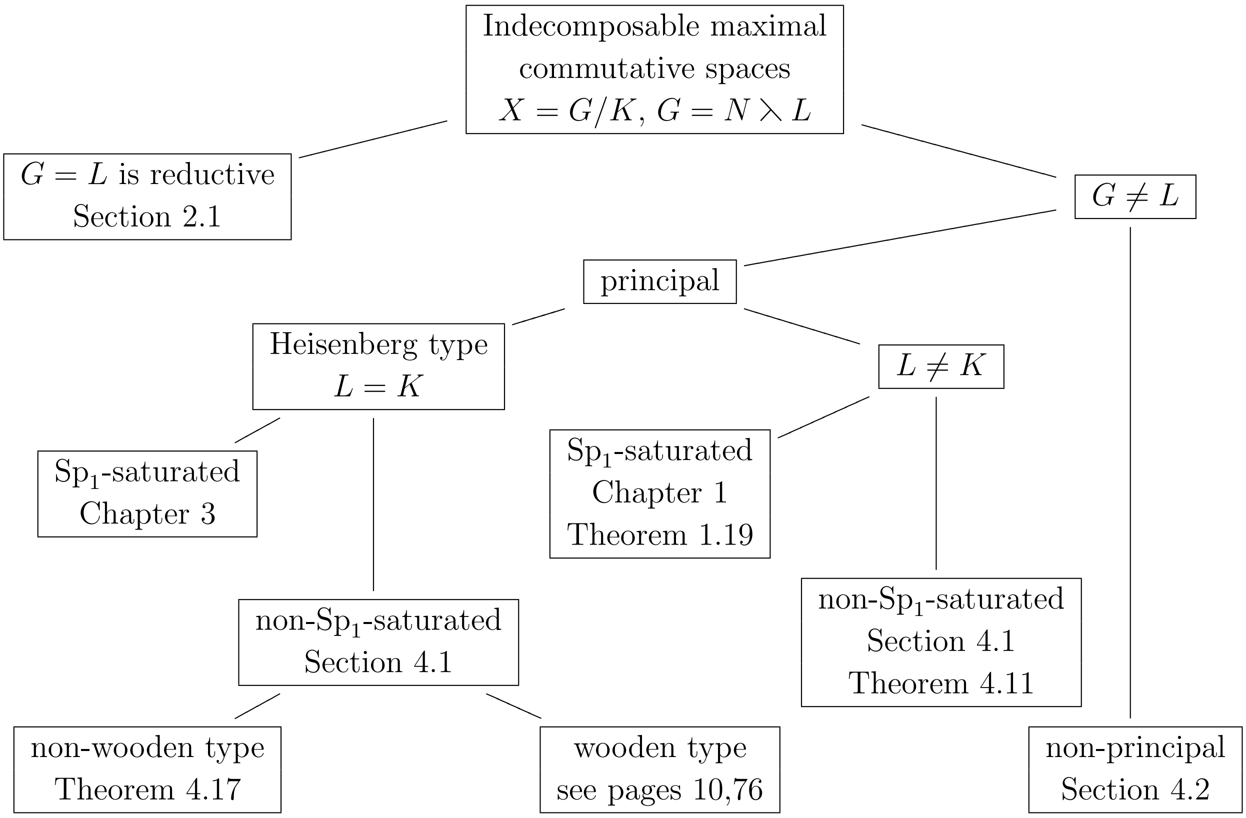
**Theorem 10.** *Suppose $\widetilde{X} = \widetilde{X}_1 \times \cdots \times \widetilde{X}_s \times \widetilde{X}_{\mathrm{Heis}} \times \widetilde{X}_{\mathrm{red}}$ is a commutative principal homogeneous space such that there is no spherical subgroups in $\widetilde{L}_{\mathrm{red}}$ between $\widetilde{K}'_{\mathrm{red}}$ and $\widetilde{K}_{\mathrm{red}}$; and $(\widetilde{N}_i \rtimes \widetilde{L}'_i)/\widetilde{K}'_i$ is never commutative. Assume that $Z(L) \subset Z(K)L'$. Then $X$ is commutative if and only if $Z(K)$ is a product $T_1 \times T_2$ such that*

$$T_1 \subset \left(\prod_{i=1}^{s} Z_i\right) \times Z_{\circledast} \times Z(\widetilde{K}_{\mathrm{red}}),\ T_2 = Z(K) \cap Z(\widetilde{K}_{\mathrm{Heis}}),\ \left(\prod_{i=1}^{s} Z_i\right) \times Z(\widetilde{K}_{\mathrm{red}}) \subset T_1 Z_{\circledast},$$

*and the action $T_2 \times \widetilde{K}'_{\mathrm{Heis}} : \widetilde{\mathfrak{n}}_{\mathrm{Heis}}$ is commutative.*

It remains to describe possible connected centres of $K$ for commutative spaces of Heisenberg type. Given $Z(K)$ we describe a simple algorithm, which allows one to check whether $X$ is commutative or not, see Lemma 4.19.

We illustrate the general classification scheme by the following diagram.



In Chapter 5, we classify principal maximal $\mathrm{Sp}_1$-saturated weakly symmetric homogeneous spaces. Recall that commutative spaces of reductive Lie groups are weakly symmetric according to [1]. Commutative spaces of Euclidian type are symmetric.

For one class of commutative spaces we prove a general statement.

**Theorem 11.** *Suppose $\mathfrak{n}$ is a direct sum of several $K$-invariant Heisenberg algebras and $X = (N \leftthreetimes K)/K$ is commutative. Then $X$ is weakly symmetric.*

For all other spaces, we check case by case whether they are weakly symmetric or not. To state the result we use notation of Table 3.2.

**Theorem 12.** *There are only eight maximal principal $\mathrm{Sp}_1$-saturated indecomposable commutative spaces, which are not weakly symmetric. They are: $((\mathbb{R}^2 \otimes \mathbb{R}^8) \leftthreetimes \mathrm{SO}_8)/\mathrm{Spin}_7$ and seven spaces of Heisenberg type with $K = \mathrm{Sp}_n$, $\mathfrak{n} = \mathbb{H}^n \oplus HS_0^2\mathbb{H}^n \oplus \mathbb{H}_0$; $K = \mathrm{Sp}_n \times \mathrm{Sp}_m$, $\mathfrak{n} = (\mathbb{H}^n \oplus \mathbb{H}_0) \oplus \mathbb{H}^n \otimes \mathbb{H}^m$; $K = \mathrm{Sp}_n$, $\mathfrak{n} = (\mathbb{H}^n \oplus \mathbb{H}_0) \oplus HS_0^2\mathbb{H}^n$; $K = \mathrm{Spin}_7 \cdot (\mathrm{SO}_2, \{E\})$, $\mathfrak{n} = (\mathbb{R}^8 \oplus \mathbb{R}^7) \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$; and $K = (\mathrm{U}_1 \cdot)\mathrm{SU}_4$, $\mathfrak{n} = (\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6 \otimes \mathbb{R}^2$. There is only one nontrivial central reduction of a maximal principal $\mathrm{Sp}_1$-saturated indecomposable commutative space, which is not weakly symmetric, namely, $(N \leftthreetimes K)/K$, where $K = \mathrm{Sp}_n$, $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}_0$.*

Let us say that $\theta \in AutG$ is a Weyl involution of $G = N \leftthreetimes L$, if $\theta(L) = L$ and $\theta|_L$ is a usual Weyl involution of $L$. (Note that the condition $\theta(N) = N$ is automatically satisfied.) We show that if $G/K$ is commutative, then a Weyl involution of $G$ exists and can be chosen such that $\theta(K) = K$. As in the reductive case, if $G/K$ is weakly symmetric, then it is weakly symmetric with respect to a Weyl involution of $G$.

To prove that $G/K$ is not weakly symmetric, we show that for any automorphism $\sigma$ of $G$ preserving $K$, there is a $K$-invariant homogeneous polynomial $f$ in $\mathbb{R}[\mathfrak{g}/\mathfrak{k}]$ such that $\sigma(f) \neq (-1)^{\deg f} f$.

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# Chapter 1

# Principal Gelfand pairs. First classification results

Let $G$ be a real Lie group, $K \subset G$ a compact subgroup and $X = G/K$.

## 1.1 A commutativity criterion

Let $U(\mathfrak{g})$ stand for the universal enveloping algebra of $\mathfrak{g}$. There is a natural filtration:

$$U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \ldots \subset U_m(\mathfrak{g}) \subset \ldots ,$$

where $U_m(\mathfrak{g}) \subset U(\mathfrak{g})$ consists of all elements of order at most $m$.

The Poisson bracket on the symmetric algebra $S(\mathfrak{g}) = \mathrm{gr}U(\mathfrak{g})$ is determined by the formula

$$\{a + U_{n-1}(\mathfrak{g}), b + U_{m-1}(\mathfrak{g})\} = [a, b] + U_{n+m-2}(\mathfrak{g}) \quad \forall a \in U_n(\mathfrak{g}), b \in U_m(\mathfrak{g}).$$

Let $X = G/K$ be a Riemannian homogeneous space. It is well known, see, for example, [43], that there is an isomorphism of the associated graded algebras:

$$\mathrm{gr}U(\mathfrak{g})^K/(U(\mathfrak{g})\mathfrak{k})^K = \mathrm{gr}\mathcal{D}(X)^G = \mathcal{P}(T^*X)^G = S(\mathfrak{g}/\mathfrak{k})^K.$$

The space $(U(\mathfrak{g})\mathfrak{k})^K$ is an ideal of $U(\mathfrak{g})^K$, also $(S(\mathfrak{g})\mathfrak{k})^K$ is a Poisson ideal of $S(\mathfrak{g})^K$. The well defined Poisson bracket on the factor $S(\mathfrak{g})^K/(S(\mathfrak{g})\mathfrak{k})^K \cong S(\mathfrak{g}/\mathfrak{k})^K$ coincides up to a sign with the Poisson bracket on $\mathcal{P}(T^*X)^G$. In particular, $X$ is commutative if and only if the Poisson algebra $S(\mathfrak{g}/\mathfrak{k})^K$ is commutative.

If $X = (N \rtimes L)/K$ is commutative, then $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$ [43]. The orbits of a compact group are separated by polynomial invariants. Hence, the last equality holds if and only if $L$ and $K$ have the same orbits in $\mathfrak{n}$. Next, a $K$-invariant positive-definite symmetric bilinear form on $\mathfrak{n}$ is automatically $L$-invariant. In particular, $\mathfrak{n}$ and $\mathfrak{n}^*$ are isomorphic as $L$-modules.

Therefore, $\mathrm{ad}^*(\mathfrak{k})\gamma = \mathrm{ad}^*(\mathfrak{l})\gamma$ for each $\gamma \in \mathfrak{n}^*$ and hence $\mathfrak{l} = \mathfrak{k} + \mathfrak{l}_\gamma$. Moreover, the natural restriction

$$\tau: \ \mathfrak{l}^* \longrightarrow \mathfrak{l}_\gamma^*$$

(which is also a homomorphism of $L_\gamma$-modules) determines an isomorphism of $K_\gamma$-modules $(\mathfrak{l}/\mathfrak{k})^*$ and $(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^*$.

Recall that $\mathfrak{g} = \mathfrak{l} + \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent ideal and $\mathfrak{l}$ is a reductive subalgebra. Let $\check{\mathfrak{n}}$ and $\check{\mathfrak{l}}$ be Abelian Lie algebras of dimensions $\dim \mathfrak{n}$ and $\dim \mathfrak{l}$, respectively. Consider new Lie algebras $\check{\mathfrak{g}}_1 = \mathfrak{l} + \check{\mathfrak{n}}$ and $\check{\mathfrak{g}}_2 = \check{\mathfrak{l}} \oplus \mathfrak{n}$, where $\check{\mathfrak{l}}$, $\check{\mathfrak{n}}$ are Abelian ideals and $\check{\mathfrak{n}} \cong \mathfrak{n}$ as an $\mathfrak{l}$-modules. That is, $\check{\mathfrak{g}}_1$ and $\check{\mathfrak{g}}_2$ are two different contractions of the Lie algebra structure on $\mathfrak{g}$.

Denote by $\{\ ,\ \}_\mathfrak{l}$ and $\{\ ,\ \}_\mathfrak{n}$ the Poisson brackets on $S(\check{\mathfrak{g}}_1)$ and $S(\check{\mathfrak{g}}_2)$. There is a $K$-invariant bi-grading $S(\mathfrak{g}) = \bigoplus S^{n,l}(\mathfrak{g})$, where $S^{n,l}(\mathfrak{g}) = S^n(\mathfrak{n})S^l(\mathfrak{l})$. We may identify $S(\mathfrak{g})$, $S(\check{\mathfrak{g}}_1)$, and $S(\check{\mathfrak{g}}_2)$ as graded commutative $\mathbb{R}$-algebras.

**Lemma 1.1.** *For any bi-homogeneous elements $\xi \in S^{n,l}(\mathfrak{g}), \eta \in S^{n',l'}(\mathfrak{g})$, we have*

$$\{\xi,\eta\} = \{\xi,\eta\}_\mathfrak{n} + \{\xi,\eta\}_\mathfrak{l}, \quad \textit{with } \{\xi,\eta\}_\mathfrak{n} \in S^{n+n'-1,l+l'}(\mathfrak{g}), \{\xi,\eta\}_\mathfrak{l} \in S^{n+n',l+l'-1}(\mathfrak{g}).$$

*In other words, the Poisson bracket on $S(\mathfrak{g})$ is a direct sum of the brackets $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$.*

*Proof.* The Poisson bracket of bi-homogeneous elements $\xi = \xi_1 ... \xi_n, \eta = \eta_1 ... \eta_m \in S(\mathfrak{g})$ is given by the formula

$$\{\xi,\eta\} = \sum_{i,j} [\xi_i, \eta_j] \xi_1 ... \widehat{\xi_i} ... \xi_n \eta_1 ... \widehat{\eta_j} ... \eta_m. \tag{1.1}$$

This expression for $\{\xi,\eta\}$ contains summands of three different types, depending on whether $\xi_i$ and $\eta_j$ are elements of $\mathfrak{l}$ or $\mathfrak{n}$. Because $[\mathfrak{l},\mathfrak{n}] \subset \mathfrak{n}$ and $\mathfrak{l}$, $\mathfrak{n}$ are subalgebras, if $\xi_i, \eta_j \in \mathfrak{n}$, then $[\xi_i,\eta_j] \in S^{n+n'-1,l+l'}(\mathfrak{g})$, otherwise $[\xi_i,\eta_j] \in S^{n+n',l+l'-1}(\mathfrak{g})$. □

In case of $\check{\mathfrak{g}}_2$, we suppose that $\check{\mathfrak{k}}$ is an Abelian subalgebra of $\check{\mathfrak{l}}$ of dimension $\dim \mathfrak{k}$. The Poisson brackets on the Poisson quotients $S(\check{\mathfrak{g}}_2/\check{\mathfrak{k}})^K = S(\check{\mathfrak{g}}_2)^K/(S(\check{\mathfrak{g}}_2)\check{\mathfrak{k}})^K$ and $S(\check{\mathfrak{g}}_1/\mathfrak{k})^K = S(\check{\mathfrak{g}}_1)^K/(S(\check{\mathfrak{g}}_1)\mathfrak{k})^K$ are still denoted by $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$, respectively. Here $\check{\mathfrak{g}}_i$, $i = 1,2$, are isomorphic to $\mathfrak{g}$ as $K$-modules. Then $\{a,b\}_\mathfrak{l} \in S^{n+n',l+l'-1}(\mathfrak{g}/\mathfrak{k})$ and $\{a,b\}_\mathfrak{n} \in S^{n+n'-1,l+l'}(\mathfrak{g}/\mathfrak{k})$ for any $a \in S^{n,l}(\mathfrak{g}/\mathfrak{k}), b \in S^{n',l'}(\mathfrak{g}/\mathfrak{k})$ $(a, b \in S(\mathfrak{g}/\mathfrak{k})^K)$.

**Lemma 1.2.** *The Poisson bracket on $S(\mathfrak{g}/\mathfrak{k})^K$ is of the form $\{\ ,\ \} = \{\ ,\ \}_\mathfrak{n} + \{\ ,\ \}_\mathfrak{l}$.*

*Proof.* This is a straightforward consequence of Lemma 1.1. □

**Corollary 1.** *Let $(N \rtimes L)/K$ be a commutative homogeneous space and $\check{N}$ a simply connected Abelian Lie group with Lie algebra $\check{\mathfrak{n}}$. Then $(\check{N} \rtimes L)/K$ is also commutative.*

Denote by $Y/\!\!/F$ the categorical quotient of an affine algebraic variety $Y$ by the action of a reductive group $F$. Set $\mathfrak{m} := \mathfrak{l}/\mathfrak{k}$.

**Theorem 1.3.** *The homogeneous space $X = (N \rightthreetimes L)/K$ is commutative if and only if all of the following three conditions hold:*

*(A) $\mathbb{R}[\mathfrak{n}]^L = \mathbb{R}[\mathfrak{n}]^K$;*

*(B) for any point $\gamma \in \mathfrak{n}^*$ the homogeneous space $L_\gamma/K_\gamma$ is commutative;*

*(C) for any point $\beta \in \mathfrak{m}^*$ the homogeneous space $(N \rightthreetimes K_\beta)/K_\beta$ is commutative.*

*Remark* 1. The statement of the theorem remains true if we replace arbitrary points by generic points in conditions (B) and (C).

*Proof.* As was already mentioned, Vinberg proved in [43] that the condition (A) holds for any commutative space. So let us assume that it is fulfilled.

Let $\gamma$ be a point in $\mathfrak{n}^*$. Recall that the $K_\gamma$-modules $\mathfrak{l}/\mathfrak{k}$ and $\mathfrak{l}_\gamma/\mathfrak{k}_\gamma$ are isomorphic. Hence, $S(\mathfrak{l}/\mathfrak{k})$ is isomorphic to $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)$ as a graded associative algebra and also as a $K_\gamma$-module.

Consider the homomorphism

$$\varphi_\gamma : \ S(\mathfrak{g}/\mathfrak{k}) \longrightarrow S(\mathfrak{g}/\mathfrak{k})/(\xi - \gamma(\xi) : \ \xi \in \mathfrak{n}) \cong S(\mathfrak{l}/\mathfrak{k}) \cong S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma).$$

Evidently, $\varphi_\gamma(S(\mathfrak{g}/\mathfrak{k})^K) \subset S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$.

Let $\xi \in \mathfrak{l}_\gamma$, $\eta \in \mathfrak{n}$. Then $\gamma(\{\xi,\eta\}) = \gamma([\xi,\eta]) = -[\mathrm{ad}^*(\xi)\gamma](\eta) = 0 = \{\xi, \gamma(\eta)\}$.

It can easily be deduced from the above statement and from the formula (1.1), that for arbitrary bi-homogeneous elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$, which can be regarded as elements of $S((\mathfrak{l}_\gamma \oplus \mathfrak{n})/\mathfrak{k}_\gamma)$, we have

$$\varphi_\gamma(\{a,b\}_{\mathfrak{l}}) = \{\varphi_\gamma(a), \varphi_\gamma(b)\},$$

where the second bracket is the Poisson bracket on $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$. In other words, $\varphi_\gamma$ is a homomorphism of the Poisson algebras $S(\check{\mathfrak{g}}_1/\mathfrak{k})^K$ and $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$.

Given $\beta \in \mathfrak{m}^*$, consider the homomorphism

$$\varphi_\beta : \ S(\mathfrak{g}/\mathfrak{k}) \longrightarrow S(\mathfrak{g}/\mathfrak{k})/(\xi - \beta(\xi) : \ \xi \in \mathfrak{m}) \cong S(\mathfrak{n}).$$

Clearly, $\varphi_\beta(S(\mathfrak{g}/\mathfrak{k})^K) \subset S(\mathfrak{n})^{K_\beta}$. Note that $\varphi_\beta$ is a homomorphism of Poisson algebras $S(\check{\mathfrak{g}}_2/\check{\mathfrak{k}})^K$ and $S(\mathfrak{n})^{K_\beta}$. For arbitrary bi-homogeneous elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$ we have

$$\varphi_\beta(\{a,b\}_{\mathfrak{n}}) = \{\varphi_\beta(a), \varphi_\beta(b)\},$$

where the second bracket is a Poisson bracket on $S(\mathfrak{n})^{K_\beta}$.

Now we show that homomorphisms $\varphi_\gamma$ and $\varphi_\beta$ are surjective. We have $S(\mathfrak{g}) = \mathbb{R}[\mathfrak{g}^*]$, $S(\mathfrak{g}/\mathfrak{k})^K = \mathbb{R}[(\mathfrak{g}/\mathfrak{k})^*]^K = \mathbb{R}[(\mathfrak{g}/\mathfrak{k})^* /\!\!/ K]$ and $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma} = \mathbb{R}[\mathfrak{m}^* /\!\!/ K_\gamma]$, $S(\mathfrak{n})^{K_\beta} = \mathbb{R}[\mathfrak{n}^* /\!\!/ K_\beta]$. Note that

$$\mathfrak{m}^* /\!\!/ K_\gamma \cong (K\gamma \times \mathfrak{m}^*) /\!\!/ K \subset (\mathfrak{g}/\mathfrak{k})^* /\!\!/ K;$$
$$\mathfrak{n}^* /\!\!/ K_\beta \cong (\mathfrak{n}^* \times K\beta) /\!\!/ K \subset (\mathfrak{g}/\mathfrak{k})^* /\!\!/ K.$$

Moreover, $K\gamma$ and $K\beta$ are closed in $\mathfrak{n}^*$ and $\mathfrak{m}^*$, respectively. Hence the subsets $(K\gamma \oplus \mathfrak{m}^*) /\!\!/ K$ and $(\mathfrak{n}^* \oplus K\beta) /\!\!/ K$ are closed in $(\mathfrak{g}/\mathfrak{k})^* /\!\!/ K$. Thus, the restrictions $\mathbb{R}[(\mathfrak{g}/\mathfrak{k})^*]^K \to \mathbb{R}[K\gamma \oplus \mathfrak{m}^*]^K$

and $\mathbb{R}[(\mathfrak{g}/\mathfrak{k})^*]^K \to \mathbb{R}[\mathfrak{n}^* \oplus K\beta]^K$ are surjective. It is therefore proved that $\varphi_\gamma$ and $\varphi_\beta$ are surjective.

($\Longleftarrow$) Suppose conditions (B) and (C) are satisfied. Clearly, $X$ is commutative if and only if both Poisson brackets $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$ equal zero on $S(\mathfrak{g}/\mathfrak{k})^K$. If $\{a,b\}_\mathfrak{l} \neq 0$ for some elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$ then there is a (generic) point $\gamma \in \mathfrak{n}^*$ such that $\varphi_\gamma(\{a,b\}_\mathfrak{l}) \neq 0$. But $\varphi_\gamma(\{a,b\}_\mathfrak{l}) = \{\varphi_\gamma(a), \varphi_\gamma(b)\} = 0$. Analogously, if $\{a,b\}_\mathfrak{n} \neq 0$ for some elements $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$, then there is a (generic) point $\beta \in \mathfrak{m}^*$ such that $\varphi_\beta(\{a,b\}_\mathfrak{l}) \neq 0$. But $\varphi_\beta(\{a,b\}_\mathfrak{l}) = \{\varphi_\beta(a), \varphi_\beta(b)\} = 0$.

($\Longrightarrow$) Suppose $X$ is commutative. Then both Poisson brackets $\{\ ,\ \}_\mathfrak{n}$ and $\{\ ,\ \}_\mathfrak{l}$ vanish on $S(\mathfrak{g}/\mathfrak{k})^K$. Hence $\{\varphi_\gamma(a), \varphi_\gamma(b)\} = 0$, $\{\varphi_\beta(a), \varphi_\beta(b)\} = 0$ for any $a, b \in S(\mathfrak{g}/\mathfrak{k})^K$. The homomorphisms $\varphi_\gamma$ and $\varphi_\beta$ are surjective, so the Poisson algebras $S(\mathfrak{l}_\gamma/\mathfrak{k}_\gamma)^{K_\gamma}$ and $S(\mathfrak{n})^{K_\beta}$ are commutative. $\square$

*Example* 1. Making use of Theorem 1.3, we verify that $(H_{2n} \rtimes \mathrm{U}_{2n})/\mathrm{Sp}_n$ is commutative. We regard $\mathrm{Lie}\, H_{2n}$ as $\mathbb{C}^{2n} \oplus \mathbb{R}$, where $\mathbb{R}$ is the centre and $\mathbb{C}^{2n}$ is the standard $\mathrm{U}_{2n}$-module.

Since $\mathrm{U}_{2n}$ and $\mathrm{Sp}_n$ are transitive on the sphere in $\mathbb{C}^{2n}$, $\mathbb{R}[\mathbb{C}^{2n}]^{\mathrm{U}_{2n}} = \mathbb{R}[q] = \mathbb{R}[\mathbb{C}^{2n}]^{\mathrm{Sp}_n}$, where $q$ is an invariant of degree 2.

The generic stabiliser for $\mathrm{Sp}_n : \mathbb{C}^{2n}$ is equal to $\mathrm{Sp}_{n-1}$. The space $\mathrm{U}_{2n-1}/\mathrm{Sp}_{n-1}$ is a compact real form of the complex spherical space $\mathrm{GL}_{2n-1}(\mathbb{C})/\mathrm{Sp}_{2n-2}(\mathbb{C})$, and, hence, is commutative.

It remains to check that condition Theorem 1.3(C) holds. Here we have $\mathfrak{m} = \mathfrak{u}_{2n}/\mathfrak{sp}_n = \bigwedge^2 \mathbb{C}^{2n}$. It is a classical result that $K_*(\Lambda\mathbb{C}^{2n}) = \underbrace{\mathrm{SU}_2 \times \ldots \times \mathrm{SU}_2}_{n}$. As a $K_*(\mathfrak{m})$-module $\mathfrak{n} = \mathfrak{v}_1 \oplus \ldots \oplus \mathfrak{v}_n \oplus \mathbb{R}$, where $\mathfrak{v}_i = \mathbb{C}^2$ for every $i$. Each $\mathfrak{v}_i$ is acted upon by its own $\mathrm{SU}_2$. Note that $[\mathfrak{v}_i, \mathfrak{v}_j] = 0$ for $i \neq j$. For $K_*(\mathfrak{m})$-invariants we have $S(\mathfrak{n})^{K_*(\mathfrak{m})} = \mathbb{R}[t_1, \ldots, t_n, \xi]$, where $t_i$ is the quadratic $\mathrm{SU}_2$-invariant in $S^2(\mathfrak{v}_i)$ and $\xi \in \mathfrak{h}'_n$. Evidently, $t_i$ and $t_j$ commute as elements of the Poisson algebra $S(\mathfrak{n})$, and $\xi$ lies in the centre of $S(\mathfrak{n})$.

## 1.2 Properties of commutative spaces

In this and subsequent sections, the commutativity criterion (Theorem 1.3) is applied to the classification problem of Gelfand pairs. As always, $X = G/K = (N \rtimes L)/K$ is a commutative homogeneous space and $P$ is the ineffective kernel of the action $L : \mathfrak{n}$. As a consequence of Theorem 1.3(A), we have $L/P \subset \mathrm{O}(\mathfrak{n})$, $L = L_*(\mathfrak{n})K$ (or, equivalently, $L/P$ is a product of $L_*(\mathfrak{n})/P$ and $K/(K \cap P)$), and $L_\gamma$ is reductive for each $\gamma \in \mathfrak{n}^*$. The latter implies that $L_\gamma/K_\gamma$ is commutative if and only if it is spherical.

We will frequently use the following result.

**Proposition 1.4.** [43, Corollaries to Proposition 10] *Let $G/K$ be commutative. Then*

*1) for any normal subgroup $Q \subset G$ the homogeneous space $G/(QK) = (G/Q)/(K/(Q \cap K))$ is commutative;*

*2) for any compact subgroup $F \subset G$ containing $K$ the homogeneous space $G/F$ is commutative;*

*3) for any subgroup $F \subset G$ containing $K$ the homogeneous space $F/K$ is commutative.*

In particular, $(G/P)/(K/(K \cap P))$ is a commutative homogeneous space of $G/P = N \rtimes (L/P)$. In this section we consider commutative spaces satisfying condition

$(*)$ $\qquad L \neq K$ and the action $L : \mathfrak{n}$ is locally effective, i.e., $P$ is finite.

This implies that $L$ is compact.

**Definition 5.** Let $M$, $F$, $G$, $K$ be Lie groups, with $F \subset M$ and $K \subset G$. The pair $(M, F)$ is called an *extension* of $(G, K)$ if

$$G \subsetneq M, \quad M = GF, \quad K = F \cap G.$$

Condition (A) means that $(L, K)$ is an extension of $(L_*(\mathfrak{n}), K_*(\mathfrak{n}))$.

Below we state and prove several properties of generic stabilisers and extensions of spherical pairs. They will be the basic classification tools.

Denote by $B(F(\mathbb{C}))$ and $U(F(\mathbb{C})) \subset B(F(\mathbb{C}))$ a Borel and a maximal unipotent subgroups of a complex reductive group $F(\mathbb{C})$.

**Lemma 1.5.** *Let a symmetric pair $(M = F \times F, F)$ with a simple compact group $F$ be an extension of a spherical pair $(G, H)$. Then $G$ contains either $F \times \{e\}$ or $\{e\} \times F$.*

*Proof.* Let $G_1$ and $G_2$ be the images of the projections of $G$ onto the first and the second factors respectively. The group $G_1 \times G_2$ acts spherically on $F \cong M/F \cong G/H$. If neither $G_1$ nor $G_2$ equals $F$, then due to [2, Theorem 4] we have $\dim B(G_i(\mathbb{C})) \leq \dim U(F(\mathbb{C}))$. Hence, $\dim B((G_1 \times G_2)(\mathbb{C})) \leq 2 \dim U(F(\mathbb{C})) < \dim F(\mathbb{C})$ and the action $(G_1 \times G_2) : F$ cannot be spherical. Assume that $G_1 = F$ but $F \times \{e\}$ is not contained in $G$. Then $G \cong F$ and $H = \{e\}$. But the pair $(F, \{e\})$ cannot be spherical. $\square$

**Lemma 1.6.** *Suppose a compact group $F \subset \mathrm{Sp}_n$ acts irreducibly on $\mathbb{H}^n$ and $F|_{\xi\mathbb{H}} = \mathrm{Sp}_1$ for generic $\xi \in \mathbb{H}^n$. Then $F = \mathrm{Sp}_n$.*

*Proof.* Let $F(\mathbb{C}) \subset \mathrm{Sp}_{2n}(\mathbb{C})$ be the complexification of $F$. Take a generic subspace $\mathbb{C}^2 \subset \mathbb{C}^{2n}$ and let $\mathrm{SL}_2 \times \mathrm{Sp}_{2n-2}$ be the subgroup of $\mathrm{Sp}_{2n}(\mathbb{C})$ preserving it. Then the intersection $F(\mathbb{C}) \cap \mathrm{SL}_2 \times \mathrm{Sp}_{2n-2}$ contains a subgroup $H \cong \mathrm{SL}_2$ acting on $\mathbb{C}^2$ non-trivially. Hence, $F(\mathbb{C})$ acts on $\mathbb{C}^{2n}$ locally transitively. It was proved by Panyushev [38] in a classification-free way, that under our assumptions $F(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C})$. $\square$

**Lemma 1.7.** *Suppose $\mathfrak{l} \subset \mathfrak{so}(V)$ is a Lie algebra. Let $\mathfrak{l}_1$ be a non-Abelian simple ideal of $\mathfrak{l}$. Denote by $\pi$ the projection onto $\mathfrak{l}_1$. If $\pi(\mathfrak{l}_*(V)) = \mathfrak{l}_1$ and $W_1$ is a non-trivial irreducible $\mathfrak{l}$-submodule of $V$ that is also non-trivial as an $\mathfrak{l}_1$-module, then $\mathfrak{l}_1 = \mathfrak{su}_2$ and $W_1$ is of the form $\mathbb{H}^1 \otimes_{\mathbb{H}} \mathbb{H}^n$, where $\mathfrak{l}$ acts on $\mathbb{H}^n$ as $\mathfrak{sp}_n$.*

*Proof.* Set $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$. We may assume that $V = W_1$. The vector space $V$ can be decomposed into a tensor product $V = V_{1,1} \otimes_{\mathbb{D}} V_1^1$ of $\mathfrak{l}_1$ and $\mathfrak{l}_2$-modules, where $\mathbb{D}$ is one of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Here $\mathfrak{l}_1$ acts trivially on $V_1^1$ and $\mathfrak{l}_2$ acts trivially on $V_{1,1}$. Both actions $\mathfrak{l}_1 : V_{1,1}$ and $\mathfrak{l}_2 : V_1^1$ are irreducible.

Let $x = x_{1,1} \otimes x_1^1 \in V$ be a non-zero decomposable vector. Because $V_{1,1}$ is a non-trivial irreducible $\mathfrak{l}_1$-module, $(\mathfrak{l}_1)_x \neq \mathfrak{l}_1$. We have $\mathfrak{l}_* \subset \mathfrak{l}_x$ up to conjugation.

Evidently, $\mathfrak{l}_x \subset \mathfrak{n}_1(x) \oplus \mathfrak{n}_2(x)$, where $\mathfrak{n}_i(x) = \{\xi \in \mathfrak{l}_i : \xi x \in \mathbb{D}x\}$. Since $\mathfrak{l}_1 = \pi(\mathfrak{l}_*) \subset \mathfrak{n}_1(x)$, we have $\mathfrak{n}_1(x) = \mathfrak{l}_1$. Hence, $\mathbb{D}x_{1,1}$ is an $\mathfrak{l}_1$-invariant subspace of $V_{1,1}$. Thus $V_{1,1} = \mathbb{D}x_{1,1}$ and $\mathfrak{l}_1 \subset \mathfrak{gl}_1(\mathbb{D})$. If $\mathbb{D}$ equals $\mathbb{R}$ or $\mathbb{C}$, $\mathfrak{gl}_1(\mathbb{D})$ is Abelian. If $\mathbb{D} = \mathbb{H}$ then $\mathfrak{l}_1(x) \subset \mathfrak{sp}_1$. Thus we have shown that $\mathfrak{l}_1 = \mathfrak{sp}_1 = \mathfrak{su}_2$ and $W_1 = \mathbb{H}^1 \otimes_{\mathbb{H}} \mathbb{H}^n$. Moreover, $\mathfrak{l}_2|_{\mathbb{H}x} = \mathfrak{sp}_1$. To conclude, notice that $\mathfrak{l}$ has to act on $\mathbb{H}^n$ as $\mathfrak{sp}_n$ by Lemma 1.6. $\square$

**Lemma 1.8.** *Suppose $\mathfrak{l}$ is an ideal of a Lie algebra $\mathfrak{f} \subset \mathfrak{so}(V)$. Denote by $\pi$ the orthogonal projection of $\mathfrak{f}$ to $\mathfrak{l}$. Then $\mathfrak{l}_*(V)$ is an ideal of $\pi(\mathfrak{f}_*(V))$ and $\pi(\mathfrak{f}_*(V))/\mathfrak{l}_*(V)$ is a direct sum of several copies of $\mathfrak{su}_2$ and an Abelian Lie ideal.*

*Proof.* Without loss of generality, we may assume that $\mathfrak{l}$ is semisimple. A generic stabiliser is defined up to conjugation. Therefore, suppose that $\mathfrak{l}_*(V) \subset \mathfrak{f}_*(V)$. Then $\mathfrak{l}_*(V) = \mathfrak{f}_*(V) \cap \mathfrak{l}$ is an ideal of $\mathfrak{f}_*(V)$. Hence, $\mathfrak{l}_*(V) = \pi(\mathfrak{l}_*(V))$ is an ideal of $\pi(\mathfrak{f}_*(V))$. Write $\mathfrak{f} = \mathfrak{l} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is the complementary ideal. Then $V$ can be represented as a direct sum

$$V = (V_1 \otimes_{\mathbb{D}_1} V^1) \oplus \ldots \oplus (V_p \otimes_{\mathbb{D}_p} V^p),$$

where $V_i$ are irreducible $\mathfrak{l}$-modules, $V^i$ are $\mathfrak{a}$-modules; $\mathfrak{l}$ (resp. $\mathfrak{a}$) acts trivially on each $V^i$ (resp. $V_i$). In each summand, the tensor product is taken over a skew-field $\mathbb{D}_i$, which equals $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ depending on $V_i$ and $V^i$. Set $\tilde{\mathfrak{l}} := \bigoplus \mathfrak{l}_i$, $\tilde{\mathfrak{a}} := \bigoplus \mathfrak{a}_i$, where

$$\begin{array}{lll} \mathfrak{l}_i = \mathfrak{so}(V_i), & \mathfrak{a}_i = \mathfrak{so}(V^i) & \text{for } \mathbb{D}_i = \mathbb{R}, \\ \mathfrak{l}_i = \mathfrak{su}(V_i), & \mathfrak{a}_i = \mathfrak{u}(V^i) & \text{for } \mathbb{D}_i = \mathbb{C}, \\ \mathfrak{l}_i = \mathfrak{sp}(V_i), & \mathfrak{a}_i = \mathfrak{sp}(V^i) & \text{for } \mathbb{D}_i = \mathbb{H}. \end{array}$$

Here $\mathfrak{sp}(V_i)$ is a *compact* real form of the corresponding complex symplectic Lie algebra. By the construction, $\mathfrak{l} \subset \tilde{\mathfrak{l}}$, $\mathfrak{a} \subset \tilde{\mathfrak{a}}$. It suffices to prove the assertion of Lemma for the larger algebra $\mathfrak{l} \oplus \tilde{\mathfrak{a}}$ in place of $\mathfrak{f}$.

Therefore we may assume without loss of generality that $\tilde{\mathfrak{a}} = \mathfrak{a}$. To make the next reduction, we set $\tilde{\mathfrak{f}} := \tilde{\mathfrak{l}} \oplus \mathfrak{a}$, and denote by $\tilde{\pi}$ the orthogonal projection of $\tilde{\mathfrak{f}}$ to $\tilde{\mathfrak{l}}$. Then $\mathfrak{l}_*(V) = \mathfrak{l} \cap \tilde{\mathfrak{l}}_*(V)$, $\pi(\mathfrak{f}_*(V)) = \mathfrak{l} \cap \tilde{\pi}(\tilde{\mathfrak{f}}_*(V))$. Hence, $\pi(\mathfrak{f}_*(V))/\mathfrak{l}_*(V) \subset \tilde{\pi}(\tilde{\mathfrak{f}}_*(V))/\tilde{\mathfrak{l}}_*(V)$. That is, it is sufficient to prove the Lemma for the pair $(\tilde{\mathfrak{f}}, \tilde{\mathfrak{l}})$.

Recall a classical result concerning generic stabilisers; namely, if $n \le m$, then

$$\begin{array}{l} (\mathrm{O}_n \times \mathrm{O}_m)_*(\mathbb{R}^n \otimes \mathbb{R}^m) = (\{\pm 1\})^n \times \mathrm{O}_{m-n}; \\ (\mathrm{U}_n \times \mathrm{U}_m)_*(\mathbb{C}^n \otimes \mathbb{C}^m) = (\mathrm{U}_1)^n \times \mathrm{U}_{m-n}; \\ (\mathrm{Sp}_n \times \mathrm{Sp}_m)_*(\mathbb{H}^n \otimes \mathbb{H}^m) = (\mathrm{Sp}_1)^n \times \mathrm{Sp}_{m-n}. \end{array}$$

Hence $\tilde{\pi}(\tilde{\mathfrak{f}}_*(V))/\tilde{\mathfrak{l}}_*(V) = \bigoplus_{i=1}^p \mathfrak{t}_i$, where $\mathfrak{t}_i$ is trivial, if $\mathbb{D}_i = \mathbb{R}$; Abelian, if $\mathbb{D}_i = \mathbb{C}$; and if $\mathbb{D}_i = \mathbb{H}$, then $\mathfrak{t}_i = \mathfrak{su}_2 \oplus \ldots \oplus \mathfrak{su}_2$ ($d_i$ times), where $d_i = \min(\dim V_i, \dim V^i)$. $\square$

Set $L_* := L_*(\mathfrak{n})$, $K_* := K_*(\mathfrak{n})$. Recall that there is a factorisation $L = Z(L) \times L_1 \times \ldots \times L_m$. Denote by $\pi_i$ the natural projection $L \to L_i$.

**Theorem 1.9.** *Suppose $G/K$ is commutative and satisfies condition $(*)$. Then any non-Abelian normal subgroup of $K$ distinct from $\mathrm{SU}_2$ is contained in a simple factor of $L$.*

*Proof.* Assume that $K_1$ is a normal subgroup of $K$ having non-trivial projections to, say, $L_1$ and $L_2$. Consider the subgroup $M = Z(L) \times \pi_1(K) \times \pi_2(K) \times L_3 \times \ldots \times L_m$. Evidently, $K \subset M$. According to Proposition 1.4, we can replace $L$ by $M$ without loss of generality or better assume from the beginning that $L_i = \pi_i(K) = \pi_i(K_1) \cong K_1$ $(i = 1, 2)$. Let $\pi_{1,2}$ denote the projection of $L$ onto $L_1 \times L_2$. By Theorem 1.3(A), $L_1 \times L_2 = K_1\pi_{1,2}(L_*)$; and by Theorem 1.3(B), $L_*/K_*$ is commutative, hence $(L_*, K_*)$ is a spherical pair. The pair $(\pi_{1,2}(L_*), \pi_{1,2}(K_*))$ is also spherical as an image of a spherical pair. Clearly, $\pi_{1,2}(K_*) \subset \pi_{1,2}(K) \cap \pi_{1,2}(L_*)$. Thus the symmetric pair $(L_1 \times L_2, K_1)$ is an extension of the spherical pair $(\pi_{1,2}(L_*), \pi_{1,2}(K) \cap \pi_{1,2}(L_*))$. By Lemma 1.5, the group $\pi_{1,2}(L_*)$ contains $L_1$ or $L_2$ (we can assume that it contains $L_1$). Then $\pi_1(L_*) = L_1$ and $L_1 = \mathrm{SU}_2$ by Lemma 1.7. $\square$

In Table 1.1, we present the list of all factorisations of compact simple Lie algebras obtained in [32].

Table 1.1.

| $\mathfrak{g}$ | $\mathfrak{g}^1$ | $\varphi^1$ | $\mathfrak{g}^2$ | $\varphi^2$ | $\mathfrak{u}$ |
|---|---|---|---|---|---|
| $\mathfrak{su}_{2n}$ | $\mathfrak{sp}_n$ | $\varpi_1$ | $\mathfrak{su}_{2n-1}$ | $\varpi_1 + \mathbf{I}$ | $\mathfrak{sp}_{n-1}$ |
| | | | $\mathfrak{su}_{2n-1} \oplus \mathbb{R}$ | | $\mathfrak{sp}_{n-1} \oplus \mathbb{R}$ |
| $\mathfrak{so}_{2n+4}$ | $\mathfrak{so}_{2n+3}$ | $\varpi_1 + \mathbf{I}$ | $\mathfrak{su}_{n+2}$ | $\varpi_1 + \varpi_{n+1}$ | $\mathfrak{su}_{n+1}$ |
| | | | $\mathfrak{su}_{n+2} \oplus \mathbb{R}$ | | $\mathfrak{su}_{n+1} \oplus \mathbb{R}$ |
| $\mathfrak{so}_{4n}$ | $\mathfrak{so}_{4n-1}$ | $\varpi_1 + \mathbf{I}$ | $\mathfrak{sp}_n$ | $\varpi_1 + \varpi_1$ | $\mathfrak{sp}_{n-1}$ |
| | | | $\mathfrak{sp}_n \oplus \mathbb{R}$ | | $\mathfrak{sp}_{n-1} \oplus \mathbb{R}$ |
| | | | $\mathfrak{sp}_n \oplus \mathfrak{su}_2$ | | $\mathfrak{sp}_{n-1} \oplus \mathfrak{su}_2$ |
| $\mathfrak{so}_{16}$ | $\mathfrak{so}_{15}$ | $\varpi_1 + \mathbf{I}$ | $\mathfrak{so}_9$ | $\varpi_4$ | $\mathfrak{so}_7$ |
| $\mathfrak{so}_8$ | $\mathfrak{so}_7$ | $\varpi_3$ | $\mathfrak{so}_7$ | $\varpi_1 + \mathbf{I}$ | $G_2$ |
| $\mathfrak{so}_7$ | $G_2$ | $\varpi_1$ | $\mathfrak{so}_5$ | $\varpi_1 + \mathbf{I} + \mathbf{I}$ | $\mathfrak{su}_2$ |
| | | | $\mathfrak{so}_5 \oplus \mathbb{R}$ | | $\mathfrak{su}_2 \oplus \mathbb{R}$ |
| | | | $\mathfrak{so}_6$ | $\varpi_1 + \mathbf{I}$ | $\mathfrak{su}_3$ |

Here $\mathfrak{g}^1$, $\mathfrak{g}^2$ are subalgebras of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^2$, $\mathfrak{u} = \mathfrak{g}^1 \cap \mathfrak{g}^2$. In all cases $n > 1$, $\varphi^1$ and $\varphi^2$ are the restrictions of the defining representation of the complexification $\mathfrak{g}(\mathbb{C})$ to $\mathfrak{g}^1(\mathbb{C})$ and $\mathfrak{g}^2(\mathbb{C})$ (whose highest weights are indicated), $\varpi_m$ are the fundamental weights, $\mathbf{I}$ is the trivial one-dimensional representation. Note that all algebras $\mathfrak{g}^1$ in Table 1.1 are simple; if $\mathfrak{g}^2$ is not simple, then $\mathfrak{g}^2 = \mathfrak{g}^3 \oplus \mathfrak{a}$, where $\mathfrak{g}^3$ is simple, $\mathfrak{a} = \mathbb{R}$ or $\mathfrak{a} = \mathfrak{su}_2$ and $\mathfrak{g} = \mathfrak{g}^1 + \mathfrak{g}^3$.

**Lemma 1.10.** *Let $\hat{L}$ be a simple non-Abelian subgroup of* $\mathrm{SO}(V)$*. Suppose $\hat{\mathfrak{l}} \neq \mathfrak{su}_2$, $V^{\hat{L}} = \{0\}$, and there are proper subgroups $\hat{K}, F \subset \hat{L}$ such that $\hat{L}_*(V)$ is a normal subgroup of $F$, $\mathfrak{f}/\hat{\mathfrak{l}}_*(V) \subset \mathfrak{su}_2$, $\hat{L} = F\hat{K}$, and the pair $(F, F \cap \hat{K})$ is spherical. Then the triple $(\hat{L}, \hat{K}, V)$ is contained in Table 1.2a.* (If $V = V_1 + V_2$ is a reducible $\hat{L}$-module, then it is assumed that each $(\hat{L}, \hat{K}, V_i)$ is also an item of Table 1.2a.)

*Proof.* By our assumptions $\hat{L} = F\hat{K}$, hence, $\hat{\mathfrak{l}} = \mathfrak{f} + \hat{\mathfrak{k}}$. Since $\hat{L}$ is simple, the factorisation $\hat{\mathfrak{l}} = \mathfrak{f} + \hat{\mathfrak{k}}$ occurs in Table 1.1. In particular, $\hat{\mathfrak{l}}$ is either $\mathfrak{su}_{2n}$ or $\mathfrak{so}_m$. Suppose $\hat{\mathfrak{l}} = \mathfrak{g}$, $\mathfrak{f} = \mathfrak{g}_i$, and $\hat{\mathfrak{k}} = \mathfrak{g}_j$, where $\{i, j\} = \{1, 2\}$. Then $\mathfrak{f} \cap \hat{\mathfrak{k}} = \mathfrak{u}$, therefore the pair $(\mathfrak{g}_i, \mathfrak{u})$ is spherical. By the hypotheses, $\hat{\mathfrak{l}}_*(V)$ is an ideal of $\mathfrak{f}$ and $\mathfrak{f}/\hat{\mathfrak{l}}_*(V) \subset \mathfrak{su}_2$. Thus $\hat{\mathfrak{l}}_*(V)$ is one of the algebras: $\mathfrak{g}_i$, $\mathfrak{g}_i'$, $\mathfrak{g}_i/\mathfrak{sp}_1$, where the last case is only possible if $\mathfrak{g}_i = \mathfrak{sp}_n \oplus \mathfrak{sp}_1$. It can be easily seen from Table 1.1, that $\hat{\mathfrak{l}}_*(V)$ is non-trivial (and even non-Abelian). Hence, the representation $\hat{\mathfrak{l}} : V$ is contained in the Élashvili's classification [14].

Suppose $\hat{\mathfrak{l}} = \mathfrak{g} = \mathfrak{su}_{2n}$. Then $\hat{\mathfrak{l}}_*(V)$ is one of the algebras: $\mathfrak{sp}_n$, $\mathfrak{su}_{2n-1}$, $\mathfrak{u}_{2n-1}$. According to [14], either $V = \mathbb{C}^{2n}$, then we obtain the first row of Table 1.2a; or $V = \mathbb{R}^6$ with $\hat{\mathfrak{l}} = \mathfrak{su}_4$, then $(\hat{L}, \hat{K}, V) = (\mathrm{SU}_4, \mathrm{U}_3, \mathbb{R}^6)$ is a particular case of item 3 up to a local isomorphism.

Suppose now that $\mathfrak{l} = \mathfrak{so}_m$. According to [14], if $m \geq 15$, then $\mathfrak{l}_*(V) = \mathfrak{so}_{m-k}$ and $V$ is the sum of $k$ copies of $\mathbb{R}^m$. It follows form Table 1.1 that $k = 1$ and $\hat{\mathfrak{l}}_*(V) = \mathfrak{f} = \mathfrak{so}_{m-1}$. According to Krämer's classification [25], the pair $(\mathfrak{f}, \mathfrak{u})$ is spherical only in one case, namely $(\mathfrak{so}_{2n+3}, \mathfrak{su}_{n+1} \oplus \mathbb{R})$. The corresponding triple is item 3 of Table 1.2a. For smaller $m$ one has to check several cases by direct computations. The result is given in rows 2a, 2b, 4a, and 4b of Table 1.2a. □

**Proposition 1.11.** *Let $(N \leftthreetimes L)/K$ be a commutative homogeneous space satisfying condition $(*)$. Suppose there is a simple direct factor $L_1 \subset L$ such that $L_1 \neq \mathrm{SU}_2$, $L_1 \not\subset K$ and $\mathfrak{n}^{L_1} \subset \mathfrak{n}'$. Then the triple $(L, K, \mathfrak{n})$ is contained in Table 1.2b.*

*Proof.* According to Theorem 1.9 $\pi_1(K) \neq L_1$. Then, by Lemma 1.7 and Theorem 1.3, there is a non-trivial factorisation $L_1 = \pi_1(L_*)\pi_1(K)$, or equivalently, $\mathfrak{l}_1 = \pi_1(\mathfrak{l}_*) + \pi_1(\mathfrak{k})$. Set $F := \pi_1(L_*)$. Due to Lemma 1.8, $(L_1)_*(\mathfrak{n}) \lhd F$ and $F/(L_1)_*(\mathfrak{n})$ is locally isomorphic to a product of $(\mathrm{U}_1)^q$ and $(\mathrm{SU}_2)^r$. It follows form Table 1.1 that $F/(L_1)_*(\mathfrak{n})$ is either finite or locally isomorphic to $\mathrm{U}_1$ or $\mathrm{SU}_2$. The pair $(F, \pi_1(K_*))$ is spherical as an image of the spherical pair $(L_*, K_*)$. Moreover, since $\pi_1(K_*) \subset F \cap \pi_1(K)$, the pair $(F, F \cap \pi_1(K))$ is also spherical. Hence the triple $(L_1, \pi_1(K), \mathfrak{n}/(\mathfrak{n}^{L_1}))$ satisfies the assumptions of Lemma 1.10 and thereby is contained in Table 1.2a.

Set $V := \mathfrak{n}/(\mathfrak{n}^{L_1})$ and let $N_{\mathrm{SO}(V)}(L_1)$ be the normaliser of $L_1$ in $\mathrm{SO}(V)$. Recall that by our hypothesis $\mathfrak{n}^{L_1} \subset \mathfrak{n}'$. Since the action $L : \mathfrak{n}$ is locally effective, the actions $L : (\mathfrak{n}/\mathfrak{n}')$ and $L : V$ are locally effective as well. Thus $L$ is contained in $\mathrm{SO}(V)$ and, hence, in $N_{\mathrm{SO}(V)}(L_1)$

up to a local isomorphism. If $(L_1, \pi_1(K), V)$ appears in row 2b, 3 or 4b of Table 1.2a, then $V$ is an irreducible orthogonal $L_1$-module and $L = L_1$, $K = \pi_1(K)$. These are items 2b, 3 and 4d of Table 1.2b.

It can be easily seen form Table 1.2a, that there at most three subgroups between $\hat{L}$ and $N_{\mathrm{SO}(V)}(\hat{L})$. Thus each triple $(\hat{L}, \hat{K}, V)$ yields at most three possibilities for $L$ and $K$. For several arising triples conditions (A) and (B) of Theorem 1.3 are not satisfied. For instance, assume that $(L_1, \pi_1(K), V)$ is a triple pointed out in row 2a of Table 1.2a and $L = \mathrm{SO}_7{\times}\mathrm{SO}_2$. Then $K \subset G_2{\times}\mathrm{SO}_2$, $(L_*)^0 = \mathrm{SO}_5$, $(K_*)^0 = \mathrm{SU}_2$. But $\mathrm{SU}_2$ is not a spherical subgroup of $\mathrm{SO}_5$. Hence, condition (B) is not satisfied. We get the same non-spherical pair $(L_*, K_*)$ in case $(L, K, V) = (\mathrm{SO}_8{\times}\mathrm{SO}_3, \mathrm{Spin}_7{\times}\mathrm{SO}_3, \mathbb{R}^8{\otimes}\mathbb{R}^3)$.

All triples $(L, K, V)$ such that $L = L_*(V)K$, $(L_*(V), K_*(V))$ is spherical and $V^{L_1} = 0$ are contained in Table 1.2b.

Now we describe possible Lie algebra structures on $\mathfrak{n}$. We claim that $\mathfrak{n}' \neq \{0\}$ only if the pair $(L, K)$ is contained in the row 1 or 4a of Table 1.2b.

Set $\mathfrak{a} := \mathfrak{n}^{L_1}$. One can identify $V$ with an $L$-invariant complement of $\mathfrak{a}$ in $\mathfrak{n}$, i.e., $\mathfrak{n} = \mathfrak{a}{\oplus}V$. Recall that by our assumptions $\mathfrak{a} \subset \mathfrak{n}'$. Hence $\mathfrak{n} = V + [V, V]$.

Note that $V$ is reducible only if $L = \mathrm{SO}_8$, $K = \mathrm{Spin}_7$. In that case $V = \mathbb{R}^8 \oplus \mathbb{R}^8$ is a direct sum of two isomorphic $L$-modules. It follows from [43, the proof of Prop. 15] that $[V, V] = 0$. Hence, $\mathfrak{n} = V$ is an Abelian Lie algebra.

Suppose now that $V$ is an irreducible $L$-module. Then $\mathfrak{n} = V \oplus \mathfrak{a}$ and $\mathfrak{a} = [V, V]$. There is an $L$-invariant surjection $\Lambda^2 V \mapsto [V, V]$. Because representation of $L$ in $\Lambda^2 V$ is completely reducible, $[V, V]$ can be regarded as an $L$-invariant subspace of $\Lambda^2 V$. In particular, $\mathfrak{a} \subset (\Lambda^2 V)^{L_1}$. The space $\Lambda^2 V$ contains non-trivial $L_1$-invariants only in cases 1 and 4a. In both these cases $\dim(\Lambda^2 V)^{L_1} = 1$ and $\mathfrak{n}$ is either an Abelian or a Heisenberg algebra. $\square$

Table 1.2a.

| | $\hat{L}$ | $\hat{K}$ | $V$ |
|---|---|---|---|
| 1 | $\mathrm{SU}_{2n}$ | $\mathrm{Sp}_n$ | $\mathbb{C}^{2n}$ |
| 2a | $\mathrm{SO}_7$ | $G_2$ | $\mathbb{R}^7 \oplus \mathbb{R}^7$ |
| 2b | $\mathrm{Spin}_7$ | $\mathrm{Spin}_6$ | $\mathbb{R}^8$ |
| 3 | $\mathrm{SO}_{2n}$ | $\mathrm{U}_n$ | $\mathbb{R}^{2n}$ |
| 4a | $\mathrm{SO}_8$ | $\mathrm{Spin}_7$ | $\mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^8$ |
| 4b | $\mathrm{SO}_8$ | $\mathrm{Sp}_2{\times}\mathrm{SU}_2$ | $\mathbb{R}^8$ |

Table 1.2b.

| | $L$ | $K$ | $\mathfrak{n}$ |
|---|---|---|---|
| 1 | $(\mathrm{S})\mathrm{U}_{2n}$ | $\mathrm{Sp}_n(\cdot\mathrm{U}_1)$ | $\mathfrak{h}_{2n}$, $\mathbb{C}^{2n}$ |
| 2a | $\mathrm{SO}_7$ | $G_2$ | $\mathbb{R}^7$ |
| 2b | $\mathrm{Spin}_7$ | $\mathrm{Spin}_6$ | $\mathbb{R}^8$ |
| 3 | $\mathrm{SO}_{2n}$ | $\mathrm{U}_n$ | $\mathbb{R}^{2n}$ |
| 4a | $\mathrm{SO}_8{\times}\mathrm{SO}_2$ | $\mathrm{Spin}_7{\times}\mathrm{SO}_2$ | $\mathfrak{h}_8, \mathbb{R}^8{\otimes}\mathbb{R}^2$ |
| 4b | $\mathrm{SO}_8$ | $\mathrm{Spin}_7$ | $\mathbb{R}^8{\otimes}\mathbb{R}^2$ |
| 4c | $\mathrm{SO}_8$ | $\mathrm{Spin}_7$ | $\mathbb{R}^8$ |
| 4d | $\mathrm{SO}_8$ | $\mathrm{Sp}_2 \times \mathrm{SU}_2$ | $\mathbb{R}^8$ |

The first row of Table 1.2b represents actually six commutative spaces. Namely, $L$ can be either $\mathrm{SU}_{2n}$ or $\mathrm{U}_{2n}$; if $L = \mathrm{U}_{2n}$, then there are two possibilities $K = \mathrm{Sp}_n$ or $\mathrm{Sp}_n \times \mathrm{U}_1$;

independently, $\mathfrak{n}$ can be either $\mathbb{C}^{2n}$ or $\mathfrak{h}_{2n}$, with $N$ being Abelian or the Heisenberg group $H_{2n}$. Similar, row 4a of Table 1.2b represents two commutative spaces. Commutativity of each item of Table 1.2b can easily be proved by means of Theorem 1.3. For the homogeneous space contained in row 1 it was done in Example 1. Consider two more examples.

*Example* 2. The homogeneous space $(\mathbb{R}_{2n} \leftthreetimes \mathrm{SO}_{2n})/\mathrm{U}_n$, indicated in row 3 of Table 1.2b, is commutative. Since $\mathfrak{n}$ is Abelian here, condition (C) of Theorem 1.3 is automatically satisfied. For condition (A), we have $\mathbb{R}[\mathbb{R}^{2n}]^{\mathrm{SO}_{2n}} = \mathbb{R}[q] = \mathbb{R}[\mathbb{R}^{2n}]^{\mathrm{U}_n}$. It is easily seen that $L_* = \mathrm{SO}_{2n-1}$ and $K_* = \mathrm{U}_{n-1}$. The corresponding homogeneous space $\mathrm{SO}_{2n-1}/\mathrm{U}_{n-1}$ is spherical by Krämer's classification [25].

*Example* 3. The homogeneous space $(H_8 \leftthreetimes (\mathrm{SO}_8{\times}\mathrm{SO}_2))/(\mathrm{Spin}_7{\times}\mathrm{SO}_2)$, indicated in row 4a of Table 1.2b, is commutative. Here $L_* = \mathrm{SO}_7$, $K_* = G_2$. The pair $(\mathrm{SO}_7, G_2)$ is spherical, see [25], and according to Table 1.1 $\mathrm{SO}_8 = G_2\mathrm{SO}_7$. It remains to check condition 1.3(C). We have $\mathfrak{m} = \mathfrak{l}/\mathfrak{k} \cong \mathbb{R}^7$, $K_*(\mathfrak{m}) = \mathrm{SU}_4{\times}\mathrm{SO}_2$, and $(\mathfrak{n}/\mathfrak{n}') \cong \mathbb{C}^4{\otimes}_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}^4{\oplus}\mathbb{C}^4$ as a $K_*(\mathfrak{m})$-module. Then $(H_8 \leftthreetimes (\mathrm{SU}_4{\times}\mathrm{SO}_2))/(\mathrm{SU}_4{\times}\mathrm{SO}_2)$ is commutative. according to [3].

**Proposition 1.12.** *Suppose $X = (N \leftthreetimes L)/K$ is an indecomposable commutative space such that $\mathfrak{n} \neq 0$, $L$ is simple, and $L \neq K$. Then $X$ is contained in Table 1.2b.*

*Proof.* The action $L : \mathfrak{n}$ is non-trivial, otherwise $X$ would be a product $N \times (L/K)$. Set $\mathfrak{a} := \mathfrak{n}^L$. According to [43, Prop. 15] $[\mathfrak{a}, \mathfrak{n}] = 0$, i.e., $\mathfrak{a}$ is an Abelian ideal of $\mathfrak{n}$. Assume that $\mathfrak{a} \not\subset \mathfrak{n}'$. Then $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{n}') \oplus \mathfrak{a}_0$ and $X = A_0 \times ((N/A_0) \leftthreetimes L)/K$, where $A_0 \subset N$, $\mathrm{Lie}\, A_0 = \mathfrak{a}_0$. Thus, $\mathfrak{a} \subset \mathfrak{n}'$ and $X$ is contained in Table 1.2b by Proposition 1.11. $\square$

Now we can partially extend Theorem 1.9 to normal subgroups $\mathrm{SU}_2 \lhd K$.

**Lemma 1.13.** *Suppose a commutative homogeneous space $(N{\leftthreetimes}L)/K$ satisfies condition $(*)$ and $K_1{\cong}\mathrm{SU}_2$ is a normal subgroup of $K$. Then either $K_1 \subset L_i$ for some direct factor $L_i \lhd L$, or $K_1$ is the diagonal of a product of at most three direct factors of $L$ isomorphic to $\mathrm{SU}_2$.*

*Proof.* Suppose $\pi_i(K_1) \neq \{e\}$ and $L_i \neq \mathrm{SU}_2$. Then $\pi_i(K) \neq L_i$. Set $\mathfrak{a} = \mathfrak{n}^{L_i}$. As we have seen in the proof of Proposition 1.11, the triple $(L_i, \pi_i(K), \mathfrak{n}/\mathfrak{a})$ satisfies conditions of Lemma 1.10, and, hence, is contained in Table 1.2a. Note that $K_1$ is a normal subgroup of $\pi_i(K)$. Thus, $L_i = \mathrm{SO}_8$, $\pi_i(K) = \mathrm{Sp}_2 \times \mathrm{SU}_2$. If $K_1$ had a non-trivial projection onto some other simple factor of $L$, then the pair $(\mathrm{SO}_8 \times \mathrm{SU}_2, \mathrm{Sp}_2 \times \mathrm{SU}_2)$ would be spherical. (Here $\mathrm{SU}_2$ is embedded in $\mathrm{SO}_8$ as the centraliser of $\mathrm{Sp}_2$ and in $\mathrm{SU}_2$ isomorphically.) But this is not the case. To conclude with, note that the pair $((\mathrm{SU}_2)^4, \mathrm{SU}_2)$ is not spherical either. $\square$

Let $G/K$ be a Gelfand pair and $(L^{\triangle}, K^{\triangle})$ a spherical subpair of $(L, K)$, i.e., $L^{\triangle} \lhd L$, $K^{\triangle} \lhd K$ and $K^{\triangle} = L^{\triangle} \cap K$. Denote by $\pi^{\triangle}$ the projection $L \to L^{\triangle}$.

**Lemma 1.14.** *If $(L^{\triangle}, K^{\triangle}) = (\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ or $(L^{\triangle}, K^{\triangle}) = (\mathrm{SU}_2, \mathrm{U}_1)$ then $\pi^{\triangle}(L_*) = L^{\triangle}$, if $(L^{\triangle}, K^{\triangle}) = (\mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ then $\pi^{\triangle}(L_*)$ equals either $L^{\triangle}$, or $\mathrm{SU}_2 \times \mathrm{U}_1$.*

*Proof.* The group $\mathrm{SU}_2$ has only trivial factorisations, besides, $(\pi^{\triangle}(L_*), \pi^{\triangle}(L_*) \cap K^{\triangle})$ is a spherical pair. In particular, $\pi^{\triangle}(L_*) \cap K^{\triangle}$ is not empty. This reasoning explains the second and the third cases. It remains to observe that in the first case the group $\pi^{\triangle}(L_*)$ can not be $\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{U}_1$, because the pair $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{U}_1, \mathrm{U}_1)$ is not spherical. □

Results of this section provide a basis for further classification of Gelfand pairs; for example, see Theorem 1.15.

## 1.3 Principal commutative spaces

Keep the previous notation. In particular, $X = G/K = (N \leftthreetimes L)/K$ is commutative, $L = Z(L) \times L_1 \times \cdots \times L_m$, and $P$ is the ineffective kernel of $L : \mathfrak{n}$. Decompose $\mathfrak{n}/\mathfrak{n}'$ into a direct sum of irreducible $L$-invariant subspaces $\mathfrak{n}/\mathfrak{n}' = \mathfrak{w}_1 \oplus \ldots \oplus \mathfrak{w}_p$.

**Definition 6.** We say that $G/K$ is *principal* if $P$ is semisimple, $Z(K) = Z(L) \times (L_1 \cap Z(K)) \times \ldots \times (L_m \cap Z(K))$ and $Z(L) = C_1 \times \ldots \times C_p$, where $C_i \subset \mathrm{GL}(\mathfrak{w}_i)$.

The condition of "principality" concerns only properties of $Z(L)$ and $Z(K)$. The classification of commutative homogeneous spaces can be divided in two parts: (1) the classification of principal commutative spaces and (2) description of possible centres of $L$ and $K$. In this Chapter, we concentrate on part (1). Part (2) is considered in Section 4.2.

Suppose $\mathfrak{z}_0 \subset [\mathfrak{n}, \mathfrak{n}]$ is an $L$-invariant subspace, and $Z_0 \subset N$ is the corresponding connected subgroup. Then $X/Z_0 = ((N/Z_0) \leftthreetimes L)/K$ is also commutative, see [43]. The passage from $X$ to $X/Z_0$ is called a *central reduction*.

**Definition 7.** A commutative homogeneous space is said to be *maximal*, if it cannot be obtained by a non-trivial central reduction from a larger one.

**Theorem 1.15.** *Let $X = (N \leftthreetimes L)/K$ be a maximal indecomposable principal commutative homogeneous space satisfying condition $(*)$. Then either $X$ is contained in Table 1.2b (and $L'$ is simple); or $(L, K)$ is isomorphic to a product of pairs* $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$, $(\mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ *or* $(\mathrm{SU}_2, \mathrm{U}_1)$ *and a pair $(K^1, K^1)$, where $K^1$ is a compact Lie group.*

*Proof.* Suppose first that each normal subgroup $L_i \neq \mathrm{SU}_2$ is contained in $K$. Then the spherical pair $(L, K)$ is a product of the "$\mathrm{SU}_2$-pairs" and $(K^1, K^1)$, where $K^1$ contains the connected centre of $L$ and all simple normal subgroups $L_i \neq \mathrm{SU}_2$.

Suppose now that there is a simple normal subgroup $L_i \neq \mathrm{SU}_2$ of $L$, which is not a subgroup of $K$. Then we prove that $X$ is contained in Table 1.2b.

Set $\mathfrak{a} = \mathfrak{n}^{L_i}$ and let $\mathfrak{n} = \mathfrak{a} \oplus \hat{V}$ be an $L$-invariant decomposition. Denote by $\hat{P}$ the identity component of the ineffective kernel of $L : \hat{V}$ and set $\hat{L} := L/\hat{P}$, $\hat{K} := K/(K \cap \hat{P})$, $\hat{n} := \hat{V} + [\hat{V}, \hat{V}]$. Then $(\hat{N} \leftthreetimes \hat{L})/\hat{K}$ is commutative by Proposition 1.4. Due to Theorem 1.9

$\pi_i(K) \neq L_i$. Hence, $L_i$, which is a simple direct factor of $\hat{L}$, is not contained in $\hat{K}$. Therefore, $(\hat{N} \leftthreetimes \hat{L})/\hat{K}$ satisfies conditions of Proposition 1.11 and is contained in Table 1.2b. We can identify $\hat{L}$ with the maximal connected subgroup of $L$ acting on $\hat{V}$ locally effectively. Then $L = \hat{L} \cdot \hat{P}$. We show that if $\hat{P}$ is non-trivial, then $X$ is either decomposable or not maximal.

Since $X$ is principal, $Z(L) = \hat{C} \times C^1$, where $\hat{C} = \mathrm{GL}(\hat{V}) \cap Z(L)$. Hence, $\hat{C} \subset \hat{L}$, $C^1 \subset \hat{P}$ and $L = \hat{L} \times \hat{P}$. Similarly, the connected centre $Z = Z(K)$ is a product $Z = Z(L) \times \hat{Z} \times Z^1$, where $\hat{Z} \subset \hat{L}'$ and $Z^1 \subset \hat{P}'$. According to Table 1.2b, $\hat{L} = \hat{C} \times L_i$. Let $K_j$ be a simple non-Abelian normal subgroup of $K$ such that $\pi_i(K_j) \neq \{e\}$. If $K_j \ncong \mathrm{SU}_2$, then $K_j \subset L_i$ by Theorem 1.9. If $K_j \cong \mathrm{SU}_2$, then $K_j \subset L_i$ by Lemma 1.13. We conclude that $\hat{K} = \hat{L} \cap K$ and $K = \hat{K} \times F$, where $F \subset \hat{P}$.

Evidently, $\mathfrak{a}$ is a subalgebra of $\mathfrak{n}$. Moreover, because different $L$-invariant summands of $\mathfrak{n}$ commute (see [43, Prop. 15]), we have $[\hat{V}, \mathfrak{a}] = 0$. Let $A \subset N$ be a connected subgroup with $\operatorname{Lie} A = \mathfrak{a}$. Recall that either $\hat{L} = L_i$ or $\hat{L} = \mathrm{U}_1 \times L_i$. Anyway, $\hat{L}$ acts on $\mathfrak{a}$ trivially.

Assume that $X$ is not contained in Table 1.2b, i.e., $X \neq (\hat{N} \leftthreetimes \hat{L})/\hat{K}$. If $[\hat{V}, \hat{V}] \subset \hat{V}$, we obtain a non-trivial decomposition $X = (\hat{N} \leftthreetimes \hat{L})/\hat{K} \times (A \leftthreetimes \hat{P})/F$. But by our assumptions $X$ is indecomposable, hence, $[\hat{V}, \hat{V}] \not\subset \hat{V}$ and $\hat{\mathfrak{n}}' \neq 0$. According to Table 1.2b, $\hat{\mathfrak{n}} = \hat{V} \oplus \mathfrak{z}$, where $\mathfrak{z} \cong \mathbb{R}$ is a trivial $L$-module. Let $\mathfrak{a}_0$ be an $L$-invariant complement of $\mathfrak{z}$ in $\mathfrak{a}$, i.e., $\mathfrak{n} = V \oplus \mathfrak{z} \oplus \mathfrak{a}_0$. If $\mathfrak{a}_0$ is a subalgebra of $\mathfrak{n}$ (then it is an ideal), we again obtain a decomposition of $X$. If $\mathfrak{z} \subset [\mathfrak{a}_0, \mathfrak{a}_0]$, then $X$ is a central reduction of $(\hat{N} \leftthreetimes \hat{L})/\hat{K} \times (A \leftthreetimes \hat{P})/F$ by a one dimensional subgroup embedded diagonally into $\hat{N}' \times A'$. Hence, $X$ is not maximal. □

Item 1 of Table 1.2b is maximal if and only if $\mathfrak{n} = \mathfrak{h}_{2n}$, and it is principal if and only if $\hat{L} = \mathrm{SU}_{2n}$, $\hat{K} = \mathrm{Sp}_n$ or $\hat{L} = \mathrm{U}_{2n}$, $\hat{K} = \mathrm{U}_1 \cdot \mathrm{Sp}_n$; item 4a of Table 1.2b is maximal if and only if $\mathfrak{n} = \mathfrak{h}_8$. Homogeneous spaces corresponding to other rows of Table 1.2b are maximal and principal.

## 1.4 The ineffective kernel

The symbols $G, L, N, K$ have the same meaning, as above. In this section we describe possible ineffective kernels $P$ of actions $L : \mathfrak{n}$. Let $L^\diamond$ be the maximal connected normal subgroup of $L$ acting on $\mathfrak{n}$ locally effectively. Then $L$ can be decomposed as $L = P \cdot L^\diamond$. We assume that $G/K$ is indecomposable and $G$ is not reductive, hence $P \neq L$. In this section we frequently use classification of spherical pairs [25], [10, 30].

**Lemma 1.16.** *Let $G/K$ be commutative. Suppose a normal subgroup $K_1 \neq \mathrm{SU}_2$ of $K$ is contained in neither $P$ nor $L^\diamond$. Then either $K_1 = \mathrm{SO}_n$ with $n \geq 5$, or $K_1 = \mathrm{SU}_n$ with $n \geq 3$; and there are simple direct factors $P_1$, $L_1^\diamond$ of $P$, $L^\diamond$ such that $K_1 \subset P_1 \times L_1^\diamond$, $P_1 \cong L_1^\diamond \cong K_1$.*

*Proof.* It can be seen from the classification of spherical pairs, that $K_1 \subset L_i \times L_j$. Suppose that $K_1 \subset P_1 \times L_1^\diamond$. The action $K_1 : \mathfrak{n}$ is non-trivial, otherwise $K_1$ would be a subgroup of $P$. Denote by $\pi_1^K$ the projection onto $K_1$ in $K$ and by $\pi_{1,1}$ the projection onto $P_1 \times L_1^\diamond$ in $L$. By Lemma 1.7, $\pi_1^K(K_*) \neq K_1$. Recall that $(L_*, K_*)$ is spherical. Hence, the pair $(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is also spherical. Note that $L_* = P \cdot L_*^\diamond(\mathfrak{n})$. Hence, $\pi_{1,1}(L_*) = P_1 \times \pi_1^\diamond(L_*)$, where $\pi_1^\diamond$ is a projection onto $L_1^\diamond$ in $L$.

We claim that $(K_1 \times \pi_1^K(K_*), \pi_1^K(K_*))$ is spherical. Without loss of generality, we can assume that $P_1 \cong L_1^\diamond \cong K_1$. If this is not the case, we replace $L$ by a smaller subgroup containing $K$, namely each of $P_1$ and $L_1^\diamond$ is replaced by a projection of $K$ onto it. We illustrate the embedding $\pi_{1,1}(K_*) \subset \pi_{1,1}(L_*)$ by the following diagram.

$$\begin{array}{ccccc}
\pi_{1,1}(L_*) & \cong & K_1 & \times & \pi_1^\diamond(L_*) \\
 & & & & \\
\pi_{1,1}(K_*) & = & & \pi_1^K(K_*) &
\end{array}$$

Because the pair $(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is spherical, $(K_1 \times \pi_1^K(K_*), \pi_1^K(K_*))$ is also spherical. According to the classification of spherical pairs, there are only two possibilities: either $K_1 = \mathrm{SO}_{n+1}$, $\pi_1^K(K_*) = \mathrm{SO}_n$; or $K_1 = \mathrm{SU}_{n+1}$, $\pi_1^K(K_*) = \mathrm{U}_n$.

Assume that either $P_1$ or $L_1^\diamond$ is larger than $K_1$. Then, according to classifications [10, 30], $(P_1 \times L_1^\diamond, \pi_{1,1}(K))$ is one of the following six pairs.

$$\begin{array}{cccccccc}
\mathfrak{su}_n & \mathfrak{su}_{n+1}(\mathfrak{su}_{n,1}) & & \mathfrak{sp}_{n+2}(\mathfrak{sp}_{n,2}) & \mathfrak{sp}_2 & \mathfrak{so}_n & & \mathfrak{so}_{n+1}(\mathfrak{so}_{n,1}) \\
 & & & & & & & \\
\mathfrak{su}_n & & \mathfrak{u}_1 & \mathfrak{sp}_n & \mathfrak{sp}_2 & & \mathfrak{so}_n &
\end{array}$$

In particular, either $P_1$ or $L_1^\diamond$ is equal to $K_1$.

Suppose first that $P_1 \cong K_1$ and $L_1^\diamond$ is larger than $K_1$. Then we get a non-trivial factorisation $L_1^\diamond = \pi_1^\diamond(L_*)\pi_1^\diamond(K)$. Moreover, $\pi_1^\diamond(L_*) \cap \pi_1^\diamond(K)$ contains either $\mathrm{SO}_n$ or $\mathrm{U}_n$, depending on $K_1$. According to Table 1.1, $(P_1 \times L_1^\diamond, \pi_{1,1}(K)) = (\mathrm{SU}_3 \times \mathrm{SU}_4, \mathrm{U}_3)$ and $\pi_1^\diamond(L_*) = \mathrm{Sp}_2$. We have $\pi_{1,1}(K_*) \subset \mathrm{Sp}_2 \cap \mathrm{U}_3 = \mathrm{Sp}_1 \times \mathrm{U}_1$. But $\mathrm{Sp}_1 \times \mathrm{U}_1$ is not a spherical subgroup of $\pi_{1,1}(L_*) = \mathrm{SU}_3 \times \mathrm{Sp}_2$. Hence, the pair $(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is not spherical. A contradiction.

Suppose now that $L_1^\diamond \cong K_1$ and $P_1$ is larger that $K_1$. Denote by $\pi_1^P$ the projection onto $P_1$. We can decompose $\pi_1^P(K)$ into a locally direct product $\pi_1^P(K) \cong F \cdot K_1$. Then $F \cdot \pi_1^K(K_*)$ should be a spherical subgroup of $P_1$. But it is not in any of six cases listed above. $\square$

*Example* 4. The homogeneous spaces $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$ and $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$, where the normal subgroups $\mathrm{SO}_n$ and $\mathrm{SU}_n$ of $K$ are diagonally embedded into $\mathrm{SO}_n \times \mathrm{SO}_n$ and $\mathrm{SU}_n \times \mathrm{SU}_n$, respectively, are commutative. We prove it for the second space. Commutativity of the first one can be proved by the same method.

We have $L_* = \mathrm{SU}_n \times \mathrm{U}_{n-1}$ and $K_* = \mathrm{U}_{n-1}$. Clearly $L = L_* K$. According to [10, 30], $L_*/K_*$ is spherical. Thus conditions (A) and (B) of Theorem 1.3 are satisfied. To check condition (C) we show that $S(\mathfrak{n})^{K_*(\mathfrak{m})}$ is commutative. Recall that $\mathfrak{m} = \mathfrak{l}/\mathfrak{k}$. Here $K_*(\mathfrak{m}) = (\mathrm{U}_1)^n$ and $\mathfrak{n} = \mathfrak{v}_1 \oplus \ldots \oplus \mathfrak{v}_n \oplus \mathbb{R}$, where $\mathfrak{v}_i = \mathbb{R}^2$ is an irreducible $K_*(\mathfrak{m})$-module for every

$1 \leq i \leq n$. Each $\mathfrak{v}_i$ is acted upon by its own $\mathrm{U}_1$. Note that $[\mathfrak{v}_i, \mathfrak{v}_j] = 0$ for $i \neq j$. For $K_*(\mathfrak{m})$-invariants we have $S(\mathfrak{n})^{K_*(\mathfrak{m})} = \mathbb{R}[t_1, \ldots, t_n, \xi]$, where $t_i$ is the quadratic $\mathrm{U}_1$-invariant in $S^2(\mathfrak{v}_i)$ and $\xi \in \mathfrak{n}'$. Evidently, $t_i$ and $t_j$ commute as elements of the Poisson algebra $S(\mathfrak{n})$, and $\xi$ lies in the centre of $S(\mathfrak{n})$.

**Theorem 1.17.** *Suppose $X = (N \rtimes L)/K$ is a maximal principal indecomposable commutative homogeneous space. Then either $X$ is one of the spaces $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$, $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ or each non-Abelian simple normal subgroup $K_1 \neq \mathrm{SU}_2$ of $K$ is contained in $P$ or $L^\diamond$.*

*Proof.* Let $K_1 \neq \mathrm{SU}_2$ be a non-Abelian simple normal subgroup of $K$ that is contained in neither $L^\diamond$ nor $P$. Then, by Lemma 1.16, either $K_1 = \mathrm{SO}_n$ or $K_1 = \mathrm{SU}_n$ and there are $P_1 \cong L_1^\diamond \cong K_1$ such that $K_1 \subset P_1 \times L_1^\diamond$. Choose an $L$-invariant decomposition $\mathfrak{n} = \mathfrak{n}^{L_1^\diamond} \oplus V$.

Consider first the case $K_1 = \mathrm{SO}_n$. Recall that $\pi_{1,1}$ stands for the projection $L \to P_1 \times L_1^\diamond$. The pair $(\pi_{1,1}(L_*), \pi_{1,1}(K_*))$ is spherical as an image of the spherical pair $(L_*, K_*)$. According to [10, 30], $\pi_{1,1}(L_*) = \mathrm{SO}_n \times \mathrm{SO}_{n-1}$. It follows that $L_*(V) = \mathrm{SO}_{n-1}$ and using [14] we obtain $V = \mathbb{R}^n$. It is easily seen that $[V, V] = 0$ and $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$ is a direct factor of $X$. But $X$ is indecomposable, and we are done.

Consider now the second case $K_1 = \mathrm{SU}_n$. Here $\pi_{1,1}(L_*) = \mathrm{SU}_n \times \mathrm{U}_{n-1}$, $L_*(V) = \mathrm{(S)U}_{n-1}$, and $V = \mathbb{C}^n$. Set $C_V = Z(L) \cap \mathrm{GL}(V)$. Since $X$ is principal, $C_V \subset K$ and $C_V$ acts trivially on $\mathfrak{n}^{L_1^\diamond} \cap \mathfrak{n}'$. Assume that $C_V$ is trivial. Then $\pi_{1,1}(L_*) = \mathrm{SU}_n \times \mathrm{SU}_{n-1}$, $\pi_{1,1}(K_*) = \mathrm{SU}_{n-1}$. But the pair $(\mathrm{SU}_n \times \mathrm{SU}_{n-1}, \mathrm{SU}_{n-1})$ is not spherical. A contradiction. Thus $C_V = \mathrm{U}_1$.

Denote by $\mathfrak{n}_1 := V + [V, V]$ the Lie subalgebra generated by $V$, and by $N_1 \subset N$ the corresponding connected subgroup. Assume that $X \neq ((N_1 \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$. Then $L = (\mathrm{U}_n \times \mathrm{SU}_n) \times F$, $K = \mathrm{U}_n \times H$, where $H \subset F$ and $F$ acts on $V$ (and hence on $\mathfrak{n}_1$) trivially. Let $\mathfrak{a} \subset \mathfrak{n}^{L_1^\diamond}$ be an $L$-invariant complement of $\mathfrak{n}_1$ in $\mathfrak{n}$. Similar to the proof of Theorem 1.15, we show that $X$ is either decomposable or not maximal. If $\mathfrak{a}$ is a subalgebra, then it is an ideal, and $X$ is decomposable. Assume that $[\mathfrak{a}, \mathfrak{a}] \not\subset \mathfrak{a}$. Since the action $L_1^\diamond : \mathfrak{a}$ is trivial $[\mathfrak{a}, \mathfrak{a}] \cap V = 0$. Thus $[V, V] \subset [\mathfrak{a}, \mathfrak{a}]$ and $X$ is not maximal.

We have proved that $X = ((N_1 \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$. Since $V$ is an irreducible $L$-module $V \cap [V, V] = \{0\}$. Hence, $[V, V]$ is a trivial $L$-module. It follows that either $\mathfrak{n}_1 = \mathbb{C}^n$ or $\mathfrak{n} = \mathfrak{h}_n$. But in case $\mathfrak{n}_1 = \mathbb{C}^n$, $X$ is a a central reduction of $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ and, therefore, is not maximal. Thus $\mathfrak{n}_1 = \mathfrak{h}_n$. □

**Proposition 1.18.** *Let $G/K$ be a maximal principal indecomposable commutative space. Suppose there is a direct factor $L_i \neq \mathrm{Sp}_1$, $L_i \not\subset P$ such that $\pi_i(K) \neq L_i$. Then $X$ is contained in Table 1.2b.*

*Proof.* The commutative space $(G/P)/(K/(K \cap P))$ is contained in Table 1.2b due to Theorem 1.15. In particular, $(L^\diamond)' = L_i$. Assume that $P$ is not trivial. Since $G/K$ is principal and indecomposable, there is a simple direct factor $K_1 \lhd K$ which is contained in neither $L^\diamond$ nor

$P$. According to Theorem 1.17, $K_1 = \mathrm{Sp}_1$. Then, as we can see from Table 1.2a, $L_i \cong \mathrm{Spin}_8$. But as was already mentioned, the pair $(\mathrm{SU}_2 \times \mathrm{Spin}_8, \mathrm{SU}_2 \times \mathrm{Sp}_2)$ is not spherical. $\square$

Let $K_1 = \mathrm{SU}_2$ be a normal subgroup of $K$. Suppose it has a non-trivial projections onto $P_1$ and $L_1 \subset L^\diamond$. If $L_1 \neq \mathrm{SU}_2$, then $\pi_1(K) \neq L_1$ and $L_1 \cong \mathrm{Spin}_8$. But the pair $(\mathrm{SU}_2 \times \mathrm{Spin}_8, \mathrm{SU}_2 \times \mathrm{Sp}_2)$ is not spherical. Thus $L_1 = \mathrm{SU}_2$.

If $\pi_1^K(K_*) \neq K_1$, i.e., $\pi_1^K(K_*)^0 = \mathrm{U}_1$, then $K_1 \subset P_1 \times L_1^\diamond$ and $P_1 = \mathrm{SU}_2$. But if $\pi_1^K(K_*) = K_1$ (and this can be the case), then $P_1$ can be larger and $K_1$ can have a non-trivial projection onto some other simple factor $P_2$ or $L_2^\diamond = \mathrm{SU}_2$.

*Example* 5. Let $\mathrm{Sp}_{m-1,1}$ be a non-compact real form of $\mathrm{Sp}_{2m}(\mathbb{C})$. Set $P := \mathrm{Sp}_{m-1,1} \times \mathrm{Sp}_l$, $L^\diamond := \mathrm{Sp}_1 \times \mathrm{Sp}_n$, $K := \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{l-1} \times \mathrm{Sp}_1 \times \mathrm{Sp}_n$ and take for $N$ an Abelian group $\mathbb{H}^n$. The inclusions and actions are illustrated by the following diagram.

$$\begin{tikzcd}
\mathrm{Sp}_{m-1,1} \arrow[d, dash] \arrow[drr, dash] & \mathrm{Sp}_l \arrow[d, dash] \arrow[dr, dash] & \mathrm{Sp}_1 \arrow[d, dash] \arrow[drr] & \mathrm{Sp}_n \arrow[d, dash] \arrow[dr] & \\
Sp_{m-1} & \mathrm{Sp}_{l-1} & \mathrm{Sp}_1 & \mathrm{Sp}_n & \mathbb{H}^n
\end{tikzcd}$$

The homogeneous space $((N \rtimes L^\diamond) \times P)/K$ is commutative. Here $L_* = \mathrm{Sp}_{m-1,1} \times \mathrm{Sp}_l \times \mathrm{Sp}_1 \times \mathrm{Sp}_{n-1}$ and $K_* = \mathrm{Sp}_{m-1} \times \mathrm{Sp}_{l-1} \times \mathrm{Sp}_1 \times \mathrm{Sp}_{n-1}$.

## 1.5 $\mathrm{Sp}_1$-saturated spaces

Keep the previous notation. Let $L_i$ be a simple direct factor of $L$. By our assumptions $L$ is a product $L = Z(L) \times L_i \times L^i$, where $L^i$ contains all direct factors $L_j$ with $j \neq i$.

**Definition 8.** A commutative homogeneous space $X$ is called $\mathrm{Sp}_1$-*saturated*, if

(1) any normal subgroup $K_1 \cong \mathrm{SU}_2$ of $K$ is contained in either $P$ or $L^\diamond$;

(2) if a simple direct factor $L_i$ is not contained in $P$ and $\pi_i(L_*) = L_i$, then $L_i \subset K$;

(3) if there is an $L$-invariant subspace $\mathfrak{w}_j \subset (\mathfrak{n}/\mathfrak{n}')$ such that for some $L_i$ the action $L_i : \mathfrak{w}_j$ is non-trivial and the action $Z(L) \times L^i : \mathfrak{w}_j$ is irreducible, then $L_i$ acts on $(\mathfrak{n}/\mathfrak{n}')/\mathfrak{w}_j$ trivially.

Note that the commutative homogeneous space described in Example 5 is not $\mathrm{Sp}_1$-saturated. Condition (1) is not fulfilled there.

*Example* 6. Set $X = ((N \rtimes (\mathrm{Sp}_n \times \mathrm{Sp}_1)) \times \mathrm{Sp}_1)/(\mathrm{Sp}_n \times \mathrm{Sp}_1)$, where $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}_0$ is a two-step nilpotent non-commutative Lie algebra with $[\mathbb{H}^n, \mathbb{H}^n] = \mathbb{H}_0$, $\mathbb{H}_0$ is the space of purely imaginary quaternions, the normal subgroup $\mathrm{Sp}_1$ of $K$ is the diagonal of the product $\mathrm{Sp}_1 \times \mathrm{Sp}_1$. Here $\mathbb{H}^n = \mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}$, where $\mathrm{Sp}_n$ acts on $\mathbb{H}^n$ and $\mathrm{Sp}_1$ acts on $\mathbb{H}^1$; $\mathbb{H}_0 \cong \mathfrak{sp}_1$ as an $L$-module, i.e., $\mathrm{Sp}_n$ acts on it trivially and $\mathrm{Sp}_1$ via adjoint representation.

Evidently, $X = (N \rtimes L)/K$ is not $\mathrm{Sp}_1$-saturated. We show that it is commutative. First we compute the generic stabiliser $L_*$. Recall that $(\mathrm{Sp}_n \times \mathrm{Sp}_1)_*(\mathbb{H}^n) = \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1$. Clearly

$L_*(\mathbb{H}_0) = \mathrm{Sp}_n \times \mathrm{U}_1 \times \mathrm{Sp}_1$, $(\mathrm{Sp}_n \times \mathrm{U}_1)_*(\mathbb{H}^n) = \mathrm{Sp}_{n-1} \times \mathrm{U}_1$. We have $L_* = \mathrm{Sp}_{n-1} \times \mathrm{U}_1 \times \mathrm{Sp}_1$, $K_* = K \cap L_* = \mathrm{Sp}_{n-1} \times \mathrm{U}_1$, $K_*(\mathfrak{m}) = \mathrm{Sp}_n \times ((\mathrm{Sp}_1)_*(\mathfrak{sp}_1)) = \mathrm{Sp}_n \times \mathrm{U}_1$. One can easily verify conditions (A) and (B) of Theorem 1.3. Tables of [43] and [44] shows that (C) is also satisfied.

If $X$ is a non-$\mathrm{Sp}_1$-saturated commutative homogeneous space, then it can be made $\mathrm{Sp}_1$-saturated by slightly enlarging $K$, $L$ and, possibly, $N$. For instance, if a simple factor $\mathrm{Sp}_1$ of $K$ has non-trivial projections onto $P$ and $L^\diamond$, then we replace $P$ by $P \times \mathrm{Sp}_1$ or $P \times \mathrm{Sp}_1 \times \mathrm{Sp}_1$ (the second replacement is needed if $\mathrm{Sp}_1$ has non-trivial projections onto two simple factors of $P$). The group $K$ is replaced by $K \times \mathrm{Sp}_1$. Starting with the commutative spaces from Example 6, we construct an $\mathrm{Sp}_1$-saturated commutative homogeneous space $(\mathrm{Sp}_1 \times \mathrm{Sp}_1/\mathrm{Sp}_1) \times (N \rtimes K/K)$, where $K = \mathrm{Sp}_1 \times \mathrm{Sp}_n$ and $N$ is the same as before.

*Example* 7. Set $L = K = \mathrm{Sp}_n \times \mathrm{Sp}_1 \times \mathrm{Sp}_m$, $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}^m \oplus \mathbb{H}_0$, where both subspaces $\mathbb{H}^n$ and $\mathbb{H}^m$ are not commutative and $[\mathbb{H}^n, \mathbb{H}^n] = [\mathbb{H}^m, \mathbb{H}^m] = \mathbb{H}_0$. We have $S(\mathfrak{g}/\mathfrak{k})^K = S(\mathfrak{n})^K = \mathbb{R}[\xi_1, \xi_2, \eta]$, where $\xi_1 \in S^2(\mathbb{H}^n)^{\mathrm{Sp}_n}$, $\xi_2 \in S^2(\mathbb{H}^m)^{\mathrm{Sp}_m}$, $\eta \in S^2(\mathbb{H}_0)^{\mathrm{Sp}_1}$, so the corresponding homogeneous space $(N \rtimes K)/K$ is commutative. The third condition of Definition 8 is not fulfilled. If we want to enlarge $L$, we also need to enlarge $N$. As an $\mathrm{Sp}_1$-saturation we get a product of two commutative spaces $(N_i \rtimes K_i)/K_i$, where $\mathfrak{n}_1 = \mathbb{H}^n \oplus \mathbb{H}_0$, $K_1 = \mathrm{Sp}_n \times \mathrm{Sp}_1$; $\mathfrak{n}_2 = \mathbb{H}^m \oplus \mathbb{H}_0$, $K_2 = \mathrm{Sp}_m \times \mathrm{Sp}_1$.

The procedure that is inverse to $\mathrm{Sp}_1$-saturation can have steps of three different types. First, one simple factor $\mathrm{Sp}_1$ of $K$ is replaced by $\mathrm{U}_1$; second, two of three simple factors $\mathrm{Sp}_1$ of $K$ are replaced by the diagonal of their product; third, several simple factors $\mathrm{Sp}_1$ of $L$ are replaced by the diagonal of their product, $K$ is replaced by the intersection of the former $K$ and new $L$ and probably $N$ is decreased.

Suppose $F \subset \mathrm{SO}(V)$, where $V$ is a finite dimensional vector space, $F = \mathrm{Sp}_1 \times \check{F}$, and $\mathbb{R}[V]^F = \mathbb{R}[V]^{\check{F}}$, i.e., $F = F_*(V)\check{F}$ or, equivalently, $F_*(V) \cong \mathrm{Sp}_1 \cdot \check{F}_*(V)$. Then one can construct several non-$\mathrm{Sp}_1$-saturated commutative homogeneous spaces, for instance, $((V \rtimes F) \times \mathrm{Sp}_1)/(\check{F} \times \mathrm{Sp}_1)$, $((V \rtimes F) \times \mathrm{Sp}_m)/(\check{F} \times \mathrm{Sp}_1 \times \mathrm{Sp}_{m-1})$, $((V \rtimes F) \times \mathrm{Sp}_{m,1})/(\check{F} \times \mathrm{Sp}_1 \times \mathrm{Sp}_m)$, $(V \rtimes F)/(\mathrm{U}_1 \times \check{F})$, where $V$ is regarded as a simply connected Abelian Lie group.

Thus, in order to classify all commutative homogeneous spaces, one should determine all such triples $(F, \check{F}, V)$. This problem is rather technical.

*Example* 8. Suppose $\check{F} = (\mathrm{Sp}_1)^n$, $V = n\mathbb{H}$ and the action $F : V$ is given by the following diagram.

$$\begin{tikzcd}[column sep=small]
\mathrm{Sp}_1 \arrow[dr, dash] & \times & \mathrm{Sp}_1 \arrow[dl, dash] \arrow[dr, dash] & \times & \mathrm{Sp}_1 \arrow[dl, dash] \arrow[dr, dash] & \times & \cdots & \times & \mathrm{Sp}_1 \arrow[dl, dash] \arrow[dr, dash] & \times & \mathrm{Sp}_1 \arrow[dl, dash] \\
 & \mathbb{H} & \oplus & \mathbb{H} & \oplus & \mathbb{H} & \oplus \ldots \oplus & \mathbb{H} & \oplus & \mathbb{H} &
\end{tikzcd}$$

Then $F_*(V) = \mathrm{Sp}_1$, $\check{F}_*(V) = \{e\}$, $F_*(V) \cong \mathrm{Sp}_1 \times \check{F}_*(V)$, and, hence, $F = F_*(V)\check{F}$.

The description of all such triples $(F, \check{F}, V)$ can be stated in terms of certain weighted trees. But the commutative spaces obtained in this way do not differ much from either reducible ones or spaces of Euclidian type. In this chapter we classify $\mathrm{Sp}_1$-saturated commutative spaces and postponed technical details until Chapter 4.

**Theorem 1.19.** *Any maximal indecomposable principal* $\mathrm{Sp}_1$*-saturated commutative homogeneous space belongs to the one of the following four classes:*

*1) affine spherical homogeneous spaces of reductive real Lie groups;*

*2) spaces corresponding to the rows of Table 1.2b;*

*3) homogeneous space* $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$*,* $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$*, where the normal subgroups* $\mathrm{SO}_n$ *and* $\mathrm{SU}_n$ *of* $K$ *are diagonally embedded into* $\mathrm{SO}_n \times \mathrm{SO}_n$ *and* $\mathrm{SU}_n \times \mathrm{SU}_n$*, respectively;*

*4) commutative homogeneous spaces of Heisenberg type.*

*Proof.* Let $X = G/K$ be a commutative homogeneous space. If $G$ is reductive, $X$ belongs to the first class. If $L = K$ then it is a space of Heisenberg type.

Assume that $G$ is not reductive and $L \neq K$. Suppose a simple factor $K_1$ of $K$ has non-trivial projections onto both $P$ and $L^\diamond$. Then due to condition (1) of the definition of $\mathrm{Sp}_1$-saturated commutative spaces, $K_1 \neq \mathrm{SU}_2$. By Theorem 1.17, $X$ belongs to the 3-d class. If all simple factors of $K$ are contained in either $P$ or $L^\diamond$, then, because $X$ is principal, $P^0/(P^0 \cap K)$ is a factor of $X$. But $X$ is indecomposable and $G$ is not reductive, so $P^0$ is trivial. Thus, $X$ satisfies condition $(*)$.

If there is a simple factor $L_1$ of $L$ such that $\pi_1(L_*) \neq K$ and $L_1 \subsetneqq K$, then, according to Theorem 1.15, $X$ is contained in the second class. If there is no such factor, then also by Theorem 1.15, $(L, K)$ is a product of pairs of the type $(\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$, $(\mathrm{SU}_2 \times \mathrm{SU}_2, \mathrm{SU}_2)$ or $(\mathrm{SU}_2, \mathrm{U}_1)$ and a pair $(K^1, K^1)$, where $K^1$ is a compact Lie group. But these pairs (except $(K^1, K^1)$) are not allowed in $Sp_1$-saturated commutative space. The second condition of the definition of $\mathrm{Sp}_1$-saturated commutative space contradicts conditions of Lemma 1.14. Thus, $L$ would be equal $K$, but this is not the case. $\square$

# Chapter 2

# Commutative spaces of reductive Lie groups

In this chapter we suppose that $G = L$. We keep all assumptions concerning $L$ and $K$, for instance, $G = Z(L) \times L_1 \times \cdots \times L_m$. Denote by $G(\mathbb{C})$ the complexification of $G$ and let $H \subset G(\mathbb{C})$ be the complexification of $K$. We use the same definitions of indecomposable and principal homogeneous spaces as was given in the real case.

Let $F$ be a complex reductive group. Strictly speaking, a subgroup $F_0 \subset F$ is a real form of $F$ if $F_0 = F^\tau$, where $\tau$ is a real structure on $F$, see [35, §1 of Chapter 5] for precise definitions and explanations. It will be convenient for us to say that $F_0$ is a real form of $F$ if $(F^\tau)^0 \subset F_0 \subset F^\tau$.

## 2.1 Classification

Note that $G/K$ is commutative if and only if $G'/K_r$, where $K_r = K/(Z(L) \cap K)$, is commutative. In this section we suppose that $G$ is semisimple.

Commutative homogeneous spaces of real reductive Lie groups are real forms of spherical affine homogeneous spaces, see, for example, [43]. Spherical affine homogeneous spaces of simple Lie groups are classified by Krämer [25], of semisimple groups by Brion [10] and Mikityuk [30], independently. Note that the paper [10] deals only with principal homogeneous spaces. In [30] one class of non-principal spherical homogeneous spaces is described. First, we give a complete classification of spherical affine homogeneous spaces. We do it on the Lie algebras level.

Take a finite set $\{(\mathfrak{g}_i(\mathbb{C}), \mathfrak{h}_i) | i = 1, \ldots, n\}$ of indecomposable principal spherical pairs such that each $\mathfrak{h}_i$ has a non-trivial centre $\mathfrak{z}_i$. It follows from classification, that the centre of $\mathfrak{h}_i$ is one dimensional. Assume that $(\mathfrak{g}_i(\mathbb{C}), \mathfrak{h}_i')$ is not spherical only for $1 \le i \le p$, where $p \le n$. Set $\mathfrak{g}(\mathbb{C}) = \bigoplus\limits_{i=1}^{n} \mathfrak{g}_i(\mathbb{C})$, $\widetilde{\mathfrak{h}} = \bigoplus\limits_{i=1}^{n} \mathfrak{h}_i$. Let $\mathfrak{z}$ be a central subalgebra of $\widetilde{\mathfrak{h}}$. Set $\mathfrak{h} := (\bigoplus\limits_{i=1}^{n} \mathfrak{h}_i') \oplus \mathfrak{z}$. Let

$\pi_{[p]}$ be the projection onto the sum of the first $p$ algebras $\mathfrak{g}_i$.

**Theorem 2.1.** *(i) The pair $(\mathfrak{g}, \mathfrak{h})$ is spherical if and only if $\pi_{[p]}(\mathfrak{z}) = \bigoplus_{i=1}^{p} \mathfrak{z}_i$.*

*(ii) It is decomposable if and only if $\mathfrak{z}$ can be represented as a sum $\mathfrak{z} = \mathfrak{z}^1 \oplus \mathfrak{z}^2$, where $\mathfrak{z}^1 \subset \bigoplus_{i \in I} \mathfrak{g}_i$, $\mathfrak{z}^2 \subset \bigoplus_{i \in J} \mathfrak{g}_i$ $(I, J \subset \{1, \ldots, n\})$ and the intersection $I \cap J$ is empty.*

*(iii) All non-principal spherical pairs are obtained as the result of the above procedure.*

*Proof.* (i) Denote by $\mathfrak{g}_{[p]}$ and $\mathfrak{z}_{[p]}$ the sums of the first $p$ algebras $\mathfrak{g}_i(\mathbb{C})$ and $\mathfrak{z}_i$, respectively. Suppose $(\mathfrak{g}(\mathbb{C}), \mathfrak{h})$ is spherical. Then $(\mathfrak{g}_{[p]}, \pi_{[p]}(\mathfrak{h}))$ is also spherical. Let $\mathfrak{b} \subset \mathfrak{g}_{[p]}$ be a Borel subalgebra such that $\mathfrak{g}_{[p]} = \pi_{[p]}(\mathfrak{h}) + \mathfrak{b}$. Clearly, $\mathfrak{b} = \bigoplus \mathfrak{b}_i$, where $\mathfrak{b}_i \subset \mathfrak{g}_i(\mathbb{C})$ is a Borel subalgebra of $\mathfrak{g}_i(\mathbb{C})$. For each $1 \le i \le p$, we have $\mathfrak{b}_i + \mathfrak{h}'_i + \mathfrak{z}_i = \mathfrak{g}_i(\mathbb{C})$. If for some $1 \le i \le p$, $\mathfrak{z}_i \subset (\mathfrak{b}_i + \mathfrak{h}'_i)$, then $\mathfrak{b}_i + \mathfrak{h}'_i = \mathfrak{g}_i(\mathbb{C})$ and $(\mathfrak{g}_i(\mathbb{C}), \mathfrak{h}'_i)$ is spherical, which is not the case. Hence, $(\mathfrak{b} + \pi_{[p]}(\mathfrak{h}')) \cap \mathfrak{z} = 0$ and $+\mathfrak{g}_{[p]} = (\mathfrak{b} + \pi_{[p]}(\mathfrak{h}')) \oplus \mathfrak{z}_{[p]}$. On the other hand, $\pi_{[p]}(\mathfrak{h}) = \pi_{[p]}(\mathfrak{h}') \oplus \pi_{[p]}(\mathfrak{z})$. Thus $\mathfrak{g}_{[p]} = (\mathfrak{b} + \pi_{[p]}(\mathfrak{h}')) \oplus \pi_{[p]}(\mathfrak{z})$ and $\pi_{[p]}(\mathfrak{z}) = \mathfrak{z}_{[p]}$.

If $\pi_{[p]}(\mathfrak{z}) = \mathfrak{z}_{[p]}$, we take a Borel subalgebra $\mathfrak{b} = \bigoplus_{i=1}^{n} \mathfrak{b}_i \subset \mathfrak{g}(\mathbb{C})$, where $\mathfrak{b}_i \subset \mathfrak{g}_i(\mathbb{C})$ are Borel subalgebras, such that $\mathfrak{g}_i(\mathbb{C}) = \mathfrak{h}_i + \mathfrak{b}_i$ for $1 \le i \le p$ and $\mathfrak{g}_i(\mathbb{C}) = \mathfrak{h}'_i + \mathfrak{b}_i$ for $p < i \le n$. Then $\mathfrak{g}(\mathbb{C}) = \mathfrak{h} + \mathfrak{b}$ and $(\mathfrak{g}(\mathbb{C}), \mathfrak{h})$ is spherical.

(ii) This statement is absolutely clear.

(iii) Let $(\mathfrak{g}(\mathbb{C}), \mathfrak{h})$ be an indecomposable non-principal spherical pair. Denote by $\widetilde{\mathfrak{h}}$ the centraliser of $\mathfrak{h}$ in $\mathfrak{g}(\mathbb{C})$. Then $(\mathfrak{g}(\mathbb{C}), \widetilde{\mathfrak{h}})$ is a principal spherical pair. Let $(\mathfrak{g}(\mathbb{C}), \widetilde{\mathfrak{h}}) = \bigoplus_{i=1}^{n} (\mathfrak{g}_i(\mathbb{C}), \mathfrak{h}_i)$ be the decomposition into the sum of indecomposable spherical pairs. Because $(\mathfrak{g}(\mathbb{C}), \mathfrak{h})$ is indecomposable, each $\mathfrak{h}_i$ has a non-trivial (one dimensional) centre $\mathfrak{z}_i$. Clearly, $\mathfrak{h} = (\bigoplus_{i=1}^{n} \mathfrak{h}'_i) \oplus \mathfrak{z}$, where $\mathfrak{z}$ is a central subalgebra of $\widetilde{\mathfrak{h}}$. $\square$

Suppose that $G/K$ is a Riemannian homogeneous space. Then $H$ is a reductive subgroup of $G(\mathbb{C})$ and $Y = G(\mathbb{C})/H$ is an affine algebraic variety. It can be easily seen, that the homogeneous space $G/K$ is commutative if and only if $Y$ is spherical, see, for example, [43].

Suppose $(G(\mathbb{C}), H)$ is a spherical pair of connected complex reductive groups. Let $K$ be compact real form of $H$. Each real form $G$ of $G(\mathbb{C})$ containing $K$ gives rise to a commutative homogeneous space $G/K$ and all of them arise in this way. The subgroup $K$ is contained in a compact real form (maximal compact subgroup) of $G(\mathbb{C})$. Non-compact connected real forms $G$ of $G(\mathbb{C})$ containing $K$ are described by the following theorem.

**Theorem 2.2.** *Suppose $G$ is a connected non-compact real form of $G(\mathbb{C})$ such that $K \subset G$. Then $G = (G(\mathbb{C})^{\tau\varphi})^0$, $K = H^{\tau}$, where $\varphi$ is an involution of $G(\mathbb{C})$ acting trivially on $H$, and $\tau$ is a compact real structure, commuting with $\varphi$ and preserving $H$.*

*Proof.* Suppose that $K$ is contained in a connected non-compact real form $G \subset G(\mathbb{C})$. Then there is a maximal compact subgroup $G^{\varphi}$, defined by an involution $\varphi$ of $G$, such that $K \subset G^{\varphi}$.

We can extend $\varphi$ to an involution of $G(\mathbb{C})$. We get $H \subset G^{\varphi}(\mathbb{C}) = G(\mathbb{C})^{\varphi} \subset G(\mathbb{C})$. The group $G^{\varphi}$ is contained in the maximal compact subgroup $G(\mathbb{C})^{\tau} \subset G(\mathbb{C})$. Clearly, $\varphi\tau = \tau\varphi$ and $G = (G(\mathbb{C})^{\tau\varphi})^0$.

On the other hand, suppose that $G = (G(\mathbb{C})^{\tau\varphi})^0$ and $H \subset G(\mathbb{C})^{\varphi}$. Set $K = H^{\tau}$. Then $K \subset G^{\varphi} \cap G(\mathbb{C})^{\tau}$, hence $K \subset G$. The subgroup $G(\mathbb{C})^{\varphi\tau}$ is determined by $\varphi$ up to the conjugation. $\square$

If $(G(\mathbb{C}), H)$ is an indecomposable symmetric pair, then $G(\mathbb{C})^{\varphi} = H$ and there is only one non-compact real form of $G(\mathbb{C})$ containing $H$. This case is well-known, see, for example, [21].

Assume that $H$ is not a symmetric subgroup of $G(\mathbb{C})$. First consider the case of simple $G(\mathbb{C})$. There are 12 non-symmetric spherical pairs $(G(\mathbb{C}), H)$ with simple $G(\mathbb{C})$, [25]. They are listed in Table 2.1.

**Theorem 2.3.** *[46, Lemmas 2,3, Theorem 3] Let $(G(\mathbb{C}), H)$ be a non-symmetric spherical pair with simple $G(\mathbb{C})$. Then all symmetric subgroups $F = G(\mathbb{C})^{\varphi}$ such that $H \subset F$ are listed in Table 2.1. All non-compact non-symmetric commutative homogeneous spaces $G/K$ of simple real groups $G$ are listed in Table 2.2 up to a local isomorphism.*

All groups in Table 2.1 are complex, all groups in Table 2.2 are real.

Table 2.1.

| | $G(\mathbb{C})$ | $H$ | $F$ |
|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathrm{SL}_k \times \mathrm{SL}_{n-k}$ | $\mathrm{S}(\mathrm{L}_k \times \mathrm{L}_{n-k})$ |
| 2 | $\mathrm{SL}_{2n+1}$ | $\mathrm{Sp}_{2n} \cdot \mathbb{C}^*$ | $\mathrm{SL}_{2n} \cdot \mathbb{C}^*$ |
| 3 | $\mathrm{SL}_{2n+1}$ | $\mathrm{Sp}_{2n}$ | $\mathrm{SL}_{2n} \cdot \mathbb{C}^*$ |
| 4 | $\mathrm{Sp}_{2n}$ | $\mathrm{Sp}_{2n-2} \times \mathbb{C}^*$ | $\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2$ |
| 5 | $\mathrm{SO}_{2n+1}$ | $\mathrm{GL}_n$ | $\mathrm{SO}_{2n}$ |
| 6 | $\mathrm{SO}_{4n+2}$ | $\mathrm{SL}_{2n+1}$ | $\mathrm{GL}_{2n+1}$ |
| 7 | $\mathrm{SO}_{10}$ | $\mathrm{Spin}_7 \times \mathrm{SO}_2$ | $\mathrm{SO}_8 \times \mathrm{SO}_2$ |
| 8 | $\mathrm{SO}_9$ | $\mathrm{Spin}_7$ | $\mathrm{SO}_8$ |
| 9 | $\mathrm{SO}_8$ | $G_2$ | $\mathrm{SO}_7$ |
| 10 | $\mathrm{SO}_7$ | $G_2$ | $-$ |
| 11 | $E_6$ | $\mathrm{Spin}_{10}$ | $\mathrm{Spin}_{10} \cdot \mathbb{C}^*$ |
| 12 | $G_2$ | $\mathrm{SL}_3$ | $-$ |

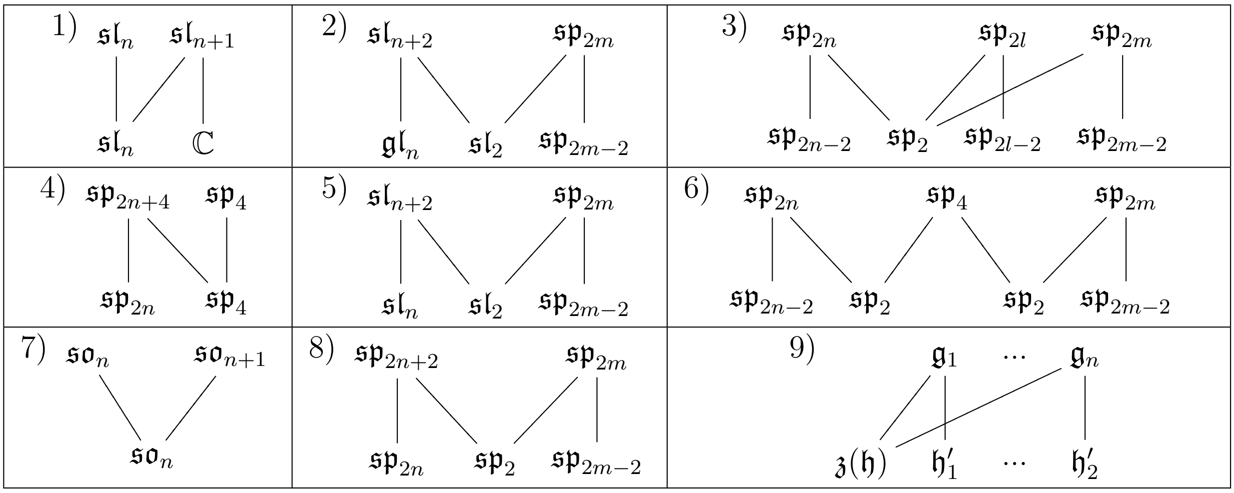
Table 2.2.

| | $G$ | $K$ |
|---|---|---|
| 1 | $\mathrm{SU}_{n,k-n}$ | $\mathrm{SU}_k \times \mathrm{SU}_{n-k}$ |
| 2 | $\mathrm{SU}_{2n,1}$ | $\mathrm{Sp}_n \cdot T$ |
| 3 | $\mathrm{SU}_{2n,1}$ | $\mathrm{Sp}_n$ |
| 4 | $\mathrm{Sp}_{n-1,1}$ | $\mathrm{Sp}_{n-1} \times T$ |
| 5 | $\mathrm{SO}^0_{2n,1}$ | $\mathrm{U}_n$ |
| 6 | $\mathrm{U}^*_{2n+1}(\mathbb{H})$ | $\mathrm{SU}_{2n+1}$ |
| 7 | $\mathrm{SO}^0_{8,2}$ | $\mathrm{Spin}_7 \times \mathrm{SO}_2$ |
| 8 | $\mathrm{SO}^0_{8,1}$ | $\mathrm{Spin}_7$ |
| 9 | $\mathrm{SO}^0_{7,1}$ | $G_2$ |
| 10 | $EIII$ | $\mathrm{Spin}_{10}$ |

Note that Table 2.2 was given in [13] without prove.

Suppose now that $G(\mathbb{C})$ is semisimple, but not simple. Then there are 8 types of non-symmetric indecomposable principal spherical pairs $(G(\mathbb{C}), H)$, see [10], [30]. We list them in Table 2.3. The case 9 describes the structure of a non-principal pair.

Table 2.3.




**Theorem 2.4.** *[46, Theorem 4] Each indecomposable non-compact non-symmetric commutative homogeneous space $G/K$ of semisimple Lie group $G$ is either of the form $K \subset K_1 \times \cdots \times K_m \subset L_1 \times \cdots \times L_m = G$, where $L_i$ are simple direct factors of $G$ and $L_i/K_i$ are commutative homogeneous spaces; or contained in Table 2.4.*

Table 2.4.

| $G$ | $K$ | $G(\mathbb{C})^{\varphi} = (G^{\varphi})^0$ | Embedding $K \subset G$ |
|---|---|---|---|
| $\mathrm{Sp}_n \times \mathrm{Sp}_2(\mathbb{C})$ | $\mathrm{Sp}_{n-1} \times \mathrm{Sp}_1$ | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_2$ | $(u, z) \to (u \oplus z, z)$ |
| $\mathrm{Sp}_{n-1,1} \times \mathrm{Sp}_2(\mathbb{C})$ | $\mathrm{Sp}_{n-1} \times \mathrm{Sp}_1$ | $\mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2 \times \mathrm{Sp}_2$ | $(u, z) \to (u \oplus z, z)$ |

## 2.2 Weakly symmetric structure

In case of reductive group $G$ the notion of commutative space coincides with a more geometrical notion of weakly symmetric space, see [1].

Let $G$ be a real Lie group and $K$ be a compact subgroup of $G$. We assume that the homogeneous space $X = G/K$ is connected. Suppose for a while that the action $G : X$ is effective, i.e., $K$ contains no nontrivial normal subgroups of $G$. Then $G$ can be regarded as a subgroup of $\mathrm{Diff}(X)$, the group of all diffeomorphisms of the manifold $X$. Let $s$ be a diffeomorphism of $X$.

**Definition 9.** The homogeneous space $X$ is called *weakly symmetric with respect to $s$*, if the following conditions are fulfilled:

$$sGs^{-1} = G, \tag{2.1}$$

$$\forall x, y \in X \ \exists g \in G : \quad gx = sy, \quad gy = sx. \tag{2.2}$$

The homogeneous space $X$ is called *weakly symmetric*, if it is weakly symmetric with respect to some diffeomorphism $s$.

Denote by $\hat{G}$ a subgroup $\langle G, s\rangle \subset \mathrm{Diff}(X)$ generated by $G$ and $s$.

The notion of weakly symmetric homogeneous space is introduced by Selberg in [41]. He assumed that $s^2 \in G$. For the sake of greater generality, we will not impose this constraint. It is worth mentioning, that all principal results of [1] and [6] were proved without this constraint. We will prove below that if $G$ is semisimple, then our definition is equivalent to Selberg's one.

One can introduce a $G$-invariant Riemannian metric on $X$, which automatically appears to be $\hat{G}$-invariant. This means that we can use results of [6].

**Definition 10.** A diffeomorphism $s \in \mathrm{Diff}(X)$ is said to be *righteous*, if conditions (2.1) and (2.2) hold for it.

The aim of this section is twofold. First we describe all righteous diffeomorphisms (isometries) of homogeneous spaces of semisimple Lie groups. After that we classify all non-symmetric weakly symmetric manifolds with reductive isometry group.

Suppose $s$ is a righteous diffeomorphism of $X$, and $g \in G$; then both $sg$ and $gs$ are righteous, too. The coset $sG$ is said to be *righteous*, if $s$ is a righteous diffeomorphism. Hence our task is to describe the righteous cosets of $\mathrm{Diff}(X)/G$.

For any $x \in X$, let $\mathrm{Diff}(X)_x$ denote the stabiliser of $x$ in $\mathrm{Diff}(X)$. It is clear that the intersection $sG \cap \mathrm{Diff}(X)_{eK}$ is not empty for every $sG$. If $s \in \mathrm{Diff}(X)_{eK}$, then $s(gK) = \sigma(g)K$, here $\sigma \in \mathrm{Aut}G$ and $\sigma(g) = sgs^{-1}$. Besides the following condition holds:

(a) $\sigma(K) = K$.

Let us denote by $\mathrm{Aut}(G,K)$ (resp. $\mathrm{Int}(G,K)$) the set of all (resp. inner) automorphisms of G satisfying (a). For any $\sigma \in \mathrm{Aut}(G,K)$, one can define an element $s \in \mathrm{Diff}(X)$ by

$$s(gK) = \sigma(g)K.$$

For any $g \in G$, let $a(g)$ denote the conjugation in $G$ by $g$. Let $\sigma, \tau$ be arbitrary elements of $\mathrm{Aut}(G,K)$. Then the corresponding diffeomorphisms of $X$ lie in the same coset in $\mathrm{Diff}(X)/G$ if and only if $\sigma = a(k)\tau$ for some $k \in K$.

The tangent space $T_{eK}X$ is canonically isomorphic to $\mathfrak{g}/\mathfrak{k}$. There is a natural action $\mathrm{Aut}(G,K) : (\mathfrak{g}/\mathfrak{k})$. If $\sigma \in \mathrm{Aut}(G,K)$ and $\eta \in \mathfrak{g}$ then $\sigma(\eta + \mathfrak{k}) := d\sigma(\eta) + \mathfrak{k}$.

It is proved in [6] that for the elements of $\mathrm{Aut}(G,K)$ condition (2.2) is equivalent to the following one:

(b) $\forall \xi \in \mathfrak{g}/\mathfrak{k}\ \exists k \in K$: $(\mathrm{Ad}k)\xi = -\sigma(\xi)$.

**Definition 11.** An automorphism $\sigma \in \mathrm{Aut}G$ is called *righteous automorphism of the pair* $(G,K)$, if the conditions (a) and (b) hold for $\sigma$.

If it does not lead to ambiguity, we will call righteous automorphisms of $(G,K)$ by righteous automorphisms of $G$. The righteous left cosets in $\mathrm{Diff}(X)/G$ are in natural one to one correspondence with the righteous left cosets in $\mathrm{Aut}(G,K)/a(K)$.

Since $K$-invariant polynomial functions on $\mathfrak{g}/\mathfrak{k}$ separate the $K$-orbits in $\mathfrak{g}/\mathfrak{k}$ (see [35]), condition (b) is equivalent to the following one:

(b') if $f \in \mathbb{R}[\mathfrak{g}/\mathfrak{k}]^K$ is a homogeneous polynomial, then $\sigma(f) = (-1)^{\deg f} f$.

**Proposition 2.5.** *([43, Lemma 1]) Let $K^0$ be the connected component of $K$, $G^0$ the connected component of $G$, and $\sigma$ a righteous automorphism of $(G, K)$. Then one can find $k \in K$ such that $a(k)\sigma$ is a righteous automorphism of $(G^0, K^0)$.*

From now on, we will consider only homogeneous spaces of connected semisimple Lie groups. Furthermore, we will assume that $K$ is connected.

Suppose now that the action $G : (G/K)$ is locally effective, but not necessarily effective. Denote by $N$ the ineffective kernel of this action, which is discrete by the definition. Let $Z(G)$ be the centre of $G$. Then $N = Z(G) \cap K$. Definition 10 can be used in this more general setting as well. Note that if $\sigma \in \mathrm{Aut}(G, K)$, then

$$\sigma(N) = \sigma(Z(G) \cap K) = \sigma(Z(G)) \cap \sigma(K) = Z(G) \cap K = N.$$

Since the action $N : (\mathfrak{g}/\mathfrak{k})$ is trivial, every righteous automorphism of $(G, K)$ defines a righteous automorphism of $(G/N, K/N)$ and a diffeomorphism of $X = G/K = (G/N)/(K/N)$. Let $\tilde{G}$ be the simply connected covering of $G$. We can consider $\mathrm{Aut}G$ and $\mathrm{Aut}(G/N)$ as subgroups of $\mathrm{Aut}\tilde{G}$. If $\mathrm{Aut}G = \mathrm{Aut}(G/N)$, then the righteous automorphisms of $(G, K)$ are the same as the righteous automorphisms of $(G/N, K/N)$.

Let $\tilde{K}$ be the connected subgroup of $\tilde{G}$ such that $\mathrm{Lie}\, K = \mathfrak{k}$. Then every righteous automorphism of $(G, K)$ lifts to a righteous automorphism of $(\tilde{G}, \tilde{K})$. We will prove now that the righteous automorphisms of those pairs are in one-to-one correspondence whenever $\mathrm{Aut}G = \mathrm{Aut}\tilde{G}(= \mathrm{Aut}\mathfrak{g})$. The group $G$ is a quotient of $\tilde{G}$ by some discrete subgroup $Z \subset Z(G)$. It is clear that $K = \tilde{K}/(\tilde{K} \cap Z)$. The actions $K : (\mathfrak{g}/\mathfrak{k})$ and $\tilde{K} : (\mathfrak{g}/\mathfrak{k})$ are the same as the action of $\tilde{K}/(\tilde{K} \cap Z(\tilde{G}))$ on $\mathfrak{g}/\mathfrak{k}$. On the other hand, suppose $\sigma$ is a righteous automorphism of $G$. Then $\sigma(K) = K$ if and only if $\sigma(\mathfrak{k}) = \mathfrak{k}$.

We consider only real groups that are real forms of simply connected complex groups. Obviously, for any semisimple Lie algebra $\mathfrak{g}$ there is only one group $G$ such that $\mathfrak{g} = \mathrm{Lie}\, G$ and $G$ satisfies this condition. We have $\mathrm{Aut}G = \mathrm{Aut}\mathfrak{g}$.

### 2.2.1 The action of $N(K)$ on $X$

Let $N(K)$ denote the normaliser of $K$ in $G$. The group $N(K)$ acts on $G/K$ by right multiplications. The group $K$ is the ineffective kernel of this action. This action commutes with the standard action of $G$. For every $n \in N(K)$ let us denote by $\Upsilon_n$ the corresponding automorphism of $X$:

$$\Upsilon_n(gK) = gKn^{-1} = gn^{-1}K.$$

**Lemma 2.6.** *Suppose $s$ is a righteous diffeomorphism of $X$, and $n \in N(K)$. Then $s\Upsilon_n s^{-1} = \Upsilon_n^{-1}$ and $s\Upsilon_n$ is righteous as well.*

*Proof.* Let us consider two points $x$ and $y = \Upsilon_n x$ of $X$. There is $g \in G$ such that:

$$gx = sy, \;\; gy = sx.$$

Then $sx = g\Upsilon_n x = \Upsilon_n(gx) = \Upsilon_n s \Upsilon_n x$. Since it is true for every $x \in X$, we have $\Upsilon_n s \Upsilon_n = s$.

Let $x, y$ be arbitrary points of $X$. By definition, there is $g \in G$ such that

$$gx = s(\Upsilon_n y), \;\; g(\Upsilon_n y) = sx.$$

Then

$$s\Upsilon_n x = \Upsilon_n^{-1} sx = \Upsilon_n^{-1}(g\Upsilon_n y) = gy, \;\; s\Upsilon_n y = gx.$$

□

**Corollary 1.** Suppose $\sigma$ is a righteous automorphism of $(G, H)$, and $n \in N(K)$. Then $\sigma(n) \equiv n^{-1} \pmod{K}$ and $a(n)\sigma$ is righteous as well.

*Proof.* Consider the corresponding diffeomorphism $s$ of $X$. We have:

$$eK = s(eK) = \Upsilon_n s \Upsilon_n(eK) = \Upsilon_n(\sigma(n^{-1})K) = \sigma(n^{-1})n^{-1}K,$$

whence $\sigma(n)n \in K$. The diffeomorphism $ns\Upsilon_n^{-1} = \Upsilon_n ns$ is righteous and stabilises $eK$. Clearly, the corresponding automorphism of $G$ is $a(n)\sigma$. □

Two righteous automorphisms $\sigma$ and $a(n)\sigma$ are said to be *equivalent*.

**Corollary 2.** The group $N(K)/K$ is Abelian and the orbits of $K$ and $N(K)$ in $\mathfrak{g}/\mathfrak{k}$ coincide.

*Proof.* It follows from Lemma 2.6 that the inversion is an automorphism of $N(K)/K$. Hence this group is abelian. □

Suppose $n \in N(K)$, and let $\sigma$ be a righteous automorphism of $G$. For $\xi \in \mathfrak{g}/\mathfrak{k}$, we then have

$$\begin{aligned}(\mathrm{Ad}n)\xi = (\mathrm{Ad}n)\sigma(\sigma^{-1}(\xi)) = -(\mathrm{Ad}k_1)\sigma^{-1}(\xi) = -(\mathrm{Ad}k_1)(-(\mathrm{Ad}k_2)\xi) = \\ = (\mathrm{Ad}(k_1k_2))\xi, \quad (k_1, k_2 \in K). \quad \square\end{aligned}$$

**Corollary 3.** The action $N(K) : \mathbb{R}[\mathfrak{g}/\mathfrak{k}]^K$ is trivial.

### 2.2.2 The complex case

We recall several results of [1] and their consequences that allow us to proceed to homogeneous spaces of complex Lie groups.

Let $G(\mathbb{C})$ be a semisimple complex algebraic group and $Y$ an indecomposable affine variety. Suppose $G$ acts transitively and effectively on $Y$, and let $s$ be an automorphism of the variety $Y$ that normalises $G$. Denote by $\hat{G}$ the subgroup $\langle G(\mathbb{C}), s\rangle \subset \mathrm{Aut}Y$. Define the action $\hat{G} : (Y \times Y)$ by:

$$s(x, y) = (sy, sx) \tag{2.3}$$

$$g(x, y) = (gx, gy). \tag{2.4}$$

**Definition 12.** The homogeneous space $Y$ is called *weakly symmetric with respect to* $s$, if the action of $s$ on algebra $\mathbb{C}[Y \times Y]^{G(\mathbb{C})}$ is trivial . The homogeneous space $Y$ is called *weakly symmetric*, if it is weakly symmetric with respect to some $s$. We call this automorphism $s$ *righteous.*

Suppose that $G/K$ is weakly symmetric (commutative) and $Y = G(\mathbb{C})/H$.

**Lemma 2.7.** *Let $N$ (resp. $N_0$) be the ineffective kernel for the action $G(\mathbb{C}) : Y$ (resp. $G : X$). If either of the groups $N$ and $N_0$ is discrete, then $N = N_0$. In particular the action $G(\mathbb{C}) : Y$ is effective if and only if the action $G : X$ is effective.*

*Proof.* If at least one of the groups $N$, $N_0$ is discrete, then both are discrete; hence they are the subgroups of the centre of $G(\mathbb{C})$ and $G$, respectively. More precisely,

$$N = Z(G(\mathbb{C})) \cap H = Z(G(\mathbb{C})) \cap Z(H), \quad N_0 = Z(G) \cap K = Z(G) \cap Z(K).$$

It is known that, $Z(K) = Z(H)$, $Z(G) = Z(G(\mathbb{C})) \cap G$. Hence,

$$N_0 = Z(G(\mathbb{C})) \cap Z(K) = Z(G(\mathbb{C})) \cap Z(H) = N.$$

□

An involution $\theta$ of a connected reductive complex algebraic group $G(\mathbb{C})$ is called a *Weyl involution*, if there exists a maximal torus of $G$, on which $\theta$ acts as inversion. It is well known that Weyl involutions exist and all such involutions are conjugate by inner automorphisms.

We will need a more precise result of [1].

**Theorem 2.8.** *([1, Theorems 2.2, 3.3, 4.2])*

*Suppose the action $G(\mathbb{C}) : Y$ is effective. Then:*

*1) Let $s$ be an automorphism of the variety $Y$ such that $sHs^{-1} = H$ and $sX = X$. Then the homogeneous space $X$ is weakly symmetric with respect to $s$ if and only if homogeneous space $Y$ is weakly symmetric with respect to $s$.*

*2) $Y$ is weakly symmetric if and only if it is spherical. More precisely, in this case it is weakly symmetric with respect to the following automorphism $t$:*

$$t(gH) = \theta(g)H,$$

*where $\theta$ is a Weyl involution of $G(\mathbb{C})$ such that $\theta(G) = G$, $\theta(H) = H$, and $\theta$ induces a Weyl involution on $H$. (In the sequel, $\theta$ always denotes a Weyl involution with these properties.)*

Automorphisms of $Y$ under consideration determine automorphisms of $G(\mathbb{C})$. Namely, to any $s \in \mathrm{Aut}Y$ one associates the automorphism $g \mapsto \sigma(g) = sgs^{-1}$ of $G(\mathbb{C})$. On the other hand, to any element of $\mathrm{Aut}(G(\mathbb{C}), H)$ one can assign an automorphism of the variety $G(\mathbb{C})/H$. Define the set of righteous automorphisms of the pair $(G(\mathbb{C}), H)$ in the same way as it was done for real groups. Similarly to the real case assume that the ineffective kernel of the action $G(\mathbb{C}) : Y$ is discrete.

In the complex case conditions (b) and (b') look as follows

(b)$_\mathbb{C}$ if $\xi \in \mathfrak{g}(\mathbb{C})/\mathfrak{h}$ and $H\xi = \overline{H\xi}$, then there is $h \in H$ such that $(\mathrm{Ad}h)\xi = -\sigma(\xi)$;

(b')$_\mathbb{C}$ if $f \in \mathbb{C}[\mathfrak{g}(\mathbb{C})/\mathfrak{h}]^H$ is a homogeneous polynomial, then $\sigma(f) = (-1)^{\deg f} f$.

Let $R$ be an irreducible representation of $G(\mathbb{C})$ and $\sigma \in \mathrm{Aut}G$. Denote by $R^\sigma = R \circ \sigma$ the twisted by $\sigma$ representation $R$ and by $R^*$ the representation dual to $R$. Suppose $G(\mathbb{C})$ acts on an affine algebraic variety $Y$. Then the $G(\mathbb{C})$-module $\mathbb{C}[Y]$ is of the form

$$\mathbb{C}[Y] = \bigoplus_R (V(R) \otimes U(R)),$$

where $R$ ranges over all irreducible representations of $G(\mathbb{C})$, and $G(\mathbb{C})$ acts via $R$ on $V(R)$ and trivially on $U(R)$.

It is well known that if $Y$ is a spherical homogeneous space of $G(\mathbb{C})$, then $\dim U(R) \leq 1$. Denote by $\mathfrak{R}$ the set of the irreducible representation $R$ such that $\dim U(R) = 1$.

**Proposition 2.9.** *Let $Y = G(\mathbb{C})/H$ be a spherical homogeneous space of $G(\mathbb{C})$ and $\mathbb{C}[Y] = \bigoplus_{R\in\mathfrak{R}} V(R)$ the decomposition of $\mathbb{C}[Y]$ into the direct sum of irreducible representations of $G(\mathbb{C})$. Let $\sigma$ be a righteous automorphism of $G(\mathbb{C}), H$. Then the following condition is satisfied*

*(c) $R^\sigma = R^*$ for every $R \in \mathfrak{R}$.*

*Proof.* Let $s$ be the automorphism of $Y$ corresponding to $\sigma$. By the definition of a righteous automorphism, $Y$ is weakly symmetric with respect to $s$. Consider the action of $G(\mathbb{C})$ on

$$\mathbb{C}[Y \times Y] = \mathbb{C}[Y] \otimes \mathbb{C}[Y] = \bigoplus_{R,S\in\mathfrak{R}} V(R) \otimes V(S).$$

As is known, each summand of the form $V(R) \otimes V(R^*)$ contains a nontrivial $G(\mathbb{C})$-invariant vector. Thus, we have

$$\forall R \in \mathfrak{R} \quad V(R) \otimes V(R^*) = s(V(R) \otimes V(R^*)) = V((R^*)^\sigma) \otimes V(R^\sigma).$$

In particular, $V(R^\sigma) = V(R^*)$. $\square$

Assume that $G(\mathbb{C})$ is simply connected. Let a homogeneous weakly symmetric space $X = G/K$ be a real form of a spherical space $Y = G(\mathbb{C})/H$ in the sense of Theorem 2.8. Let

$\sigma$ be a righteous automorphism of $G$. It extends to an automorphism of $G(\mathbb{C})$ normalising $H$. Hence $\sigma$ defines an automorphism of $Y$. In view of Theorem 2.8, this means that there is a one-to-one correspondence between the set of the righteous automorphisms of $G$ and the set of the righteous automorphisms of $G(\mathbb{C})$ satisfying the condition

(d) $\sigma(G) = G$.

If condition (d) holds, then $\sigma(K) = K$ if and only if $\sigma(H) = H$.

The action $N_G(K) : \mathfrak{g}/\mathfrak{k}$ is a real form of the action $N(H) : \mathfrak{g}(\mathbb{C})/\mathfrak{h}$. Hence the invariants of the groups $N(H)$ and $H$ in $\mathbb{C}[\mathfrak{g}(\mathbb{C})/\mathfrak{h}]$ coincide just like in the real case. In particular, if one multiplies a righteous automorphism of $(G(\mathbb{C}), H)$ by $a(n)$, where $n \in N(H)$, it will remain righteous. For $n \in N(H)$, the righteous automorphisms $\sigma$ and $a(n)\sigma$ are said to be equivalent.

Let $(G(\mathbb{C}), H)$ be a spherical pair (here $G(\mathbb{C})$ and $H$ are connected complex reductive groups). Assume that $G(\mathbb{C})$ is simple connected and semisimple. In this subsection we describe righteous automorphisms of $(G(\mathbb{C}), H)$. In each connected component of $\mathrm{Aut}G(\mathbb{C}) = \mathrm{Aut}\mathfrak{g}(\mathbb{C})$ we point out one righteous automorphism if, of course, it exists. Thus we classify all righteous automorphism up to equivalence.

**Theorem 2.10.** *Let $(G(\mathbb{C}), H)$ be a locally effective spherical pair, where $G(\mathbb{C})$ is semisimple and $H$ is reductive. Suppose $\sigma \in \mathrm{Aut}(G(\mathbb{C}), H)$ satisfies condition (c). Then $\sigma$ is the righteous automorphism of $(G(\mathbb{C}), H)$.*

*Proof.* It is sufficient to show that $\sigma$ is equivalent to some righteous automorphism. The set $\sigma\mathrm{Int}(G(\mathbb{C}), H)$ is the union of some connected components of the algebraic group $\mathrm{Aut}(G(\mathbb{C}), H)$. Hence it contains an element of finite order. We can assume that the order of $\sigma$ is finite. Assume also that the action $G(\mathbb{C}) : (G(\mathbb{C})/H)$ is effective, i.e., $G(\mathbb{C}) \subset \mathrm{Aut}Y$.

The automorphism $\sigma$ induces the automorphism $s(gH) = \sigma(g)H$ of $Y$. Clearly, the order of $s$ is finite as well. Hence the subgroup $\hat{G} = \langle G(\mathbb{C}), s\rangle \subset \mathrm{Aut}Y$ generated by $G(\mathbb{C})$ and $s$ is reductive. The variety $Y$ is the spherical homogeneous space of $\hat{G}$. More precisely, $Y = \hat{G}/\hat{H}$, where $\hat{H} = \langle H, s\rangle$.

Let $\mathbb{C}[Y] = \bigoplus\limits_{R\in\mathfrak{R}} V(R)$ be the decomposition of $\mathbb{C}[Y]$ into the direct sum of irreducible representations of $G(\mathbb{C})$. By (c) we have $s(V(R)) = V(R^*)$. Let us decompose $\mathbb{C}[Y]$ into the sum of irreducible representations of $\hat{G}$:

$$\mathbb{C}[Y] = \bigoplus_{R\in\mathfrak{R}, R=R^*} V(R) \oplus \bigoplus_{R\in\mathfrak{R}, R\neq R^*} (V(R) \oplus V(R^*)).$$

Denote by $W(R)$ the summand $V(R) \oplus V(R^*)$ from the second part of the above sum. (In fact the sum in the second part ranges over unordered pairs $\{R, R^*\}$. But we admit this formal inaccuracy for simplicity of notation.)

The representations of $\hat{G}$ in $V(R)$ and $W(R)$ are self-dual. For instance, if the representation $\hat{G} : W(R)$ is not self-dual, then the representation $\hat{G} : W(R)^*$ would occur in $\mathbb{C}[Y]$ as well. But the representations of $G(\mathbb{C})$ in $W(R)$ and $W(R)^*$ are isomorphic. Since $G(\mathbb{C})/H$ is spherical, this would lead to a contradiction.

Let us show that those representations are orthogonal. Suppose $\hat{G} : U$ is an irreducible symplectic representation of $\hat{G}$. Then there exists a non-degenerate skew-symmetric form on $U^{\hat{H}}$. It is the restriction of such form on $U$. This shows that $\dim U^{\hat{H}}$ is even. Hence such a representation cannot occur in $\mathbb{C}[Y]$.

Consider the action $\hat{G} : (Y \times Y)$ determined by (2.3) and (2.4). We have to prove that $\mathbb{C}[Y \times Y]^{G(\mathbb{C})} = \mathbb{C}[Y \times Y]^{\hat{G}}$. Recall that

$$\mathbb{C}[Y \times Y]^{G(\mathbb{C})} = \bigoplus_{R \in \mathfrak{R}, R = R^*} S^2 V(R)^{G(\mathbb{C})} \oplus \bigoplus_{R \in \mathfrak{R}, R \neq R^*} (S^2 W(R)^{G(\mathbb{C})} \oplus \Lambda^2 W(R)^{G(\mathbb{C}))},$$

besides $\dim S^2 V(R)^{G(\mathbb{C})} = \dim S^2 W(R)^{G(\mathbb{C})} = \dim \Lambda^2 W(R)^{G(\mathbb{C})} = 1$.

As is already known, there are nonzero $\hat{G}$-invariant vectors in $S^2 V(R)$ and $S^2 W(R)$, i.e., $\dim S^2 V(R)^{\hat{G}} = \dim S^2 W(R)^{\hat{G}} = 1$. In addition we have $\dim S^2 V(R)^{G(\mathbb{C})} = \dim S^2 W(R)^{G(\mathbb{C})} = 1$ as well, hence $S^2 V(R)^{\hat{G}} = S^2 V(R)^{G(\mathbb{C})}$ and $S^2 W(R)^{\hat{G}} = S^2 W(R)^{G(\mathbb{C})}$. Since the permutation $(x, y) \longrightarrow (y, x)$ acts on the spaces $S^2 V(R)$ and $S^2 W(R)$ trivially, $s$ acts trivially on $\bigoplus S^2 V(R)^{G(\mathbb{C})} \oplus \bigoplus S^2 W(R)^{G(\mathbb{C})}$.

Consider the action of $s$ on other $G(\mathbb{C})$-invariants. Let $\omega$ be an arbitrary element of $\Lambda^2 W(R)^{G(\mathbb{C})}$.

The automorphism $s^2$ normalises every representation of type $V(R)$. Consider the group $\langle G(\mathbb{C}), s^2 \rangle \subset \hat{G}$. It is reductive as well as $\hat{G}$. Since it has two linearly independent invariant vectors in $W(R) \otimes W(R)$, the form $\omega$ is $\langle G(\mathbb{C}), s^2 \rangle$-invariant. Hence $s\omega = \pm\omega$. Assume that $s\omega = -\omega$. Then $\omega$ is invariant with respect to the diagonal action $s : (Y \times Y)$ defined by $s(x, y) = (sx, sy)$. This would imply that $\dim(W(R) \otimes W(R))^{G(\mathbb{C})} = 2$ for the diagonal action of $\hat{G}$ on $Y \times Y$. But this is not the case. In fact, as we have seen, this dimension is equal to 1. $\square$

**Corollary 1.** *The set of all righteous automorphism of the pair $(G(\mathbb{C}), H)$ coincides with the union of connected components of $\mathrm{Aut}(G(\mathbb{C}), H)$ satisfying condition (c).*

**Corollary 2.** *Each closed $H$-orbits in $\mathfrak{g}(\mathbb{C})/\mathfrak{h}$ is centrally-symmetric with respect to the origin if and only if all irreducible representations occurring in the decomposition of $\mathbb{C}[G(\mathbb{C})/H]$ are self-dual.*

*Proof.* Both claims are equivalent to the fact that the identity mapping is the righteous automorphism of $(G(\mathbb{C}), H)$. $\square$

Let $B$ be a Borel subgroup of $G(\mathbb{C})$. The set of weights in $\mathbb{C}[Y]^{(B)}$ (resp. $\mathbb{C}(Y)^{(B)}$) is called the rank semigroup (resp. group) of homogeneous space $Y$. Denote by $\Gamma(Y)$ the rank group of $Y$. In each case, it would be sufficient to find the group $\mathrm{Aut}(G(\mathbb{C}), H)$ and the rank semigroup (or even the rank group) in order to describe all righteous automorphism of $(G(\mathbb{C}), H)$.

This problem can be solved by a method of D. Panyushev, see [36]. Let $T \subset B$ be a maximal torus and $\theta$ a Weyl involution of $G(\mathbb{C})$, acting on $T$ by inversion. We can replace $H$ by a conjugated group such that

I) both groups $B \cap H$ and $B \cap \theta(H)$ have the minimum possible dimensions;

II) they are both preserved by $T$;

III) $B \cap H$ is a Borel subgroup of the reductive group $H \cap \theta(H)$.

If $H$ satisfies conditions I)-III), then $H_* = H \cap \theta(H)$ is a generic stabiliser for the action $H : \mathfrak{g}/\mathfrak{h}$, and $\Gamma(G(\mathbb{C})/H)$ coincides with the annihilator of $H_* \cap T$.

Note that rank groups of homogeneous spaces of simple groups are already well known and can be found in [25]. In all other cases there are easier methods of finding the righteous automorphisms.

Let $(G(\mathbb{C}), H)$ be a spherical pair with semisimple $G(\mathbb{C})$.

**Lemma 2.11.** *Suppose $G(\mathbb{C}) = G(\mathbb{C})_1 \times \ldots \times G(\mathbb{C})_n, H = H_1 \times \ldots \times H_n$, where $(G(\mathbb{C})_i, H_i)$ is indecomposable for every $i$ and $\sigma$ is a righteous automorphism of $(G(\mathbb{C}), H)$. Then $\sigma(G(\mathbb{C})_i) = G(\mathbb{C})_i$.*

*Proof.* If $\sigma$ takes simple direct factor $F_i \lhd G(\mathbb{C})_i$ of $G(\mathbb{C})_i$ into simple direct factor $F_j \lhd G(\mathbb{C})_j$ of $G(\mathbb{C})_j$ for some $i \neq j$, then $\sigma$ takes $H_i$ into $H_j$ (otherwise $\sigma(H) \neq H$) and $G(\mathbb{C})_i$ into $G(\mathbb{C})_j$. In the decomposition of $\mathbb{C}[Y]$ into irreducibles exists a representation, say $R$, such that $G(\mathbb{C})_k$ acts trivially on $V(R)$ for each $k \neq i$. Then $G(\mathbb{C})_j$ acts trivially on the space $V(R^*)$ and non-trivially on $V(R^\sigma)$. A contradiction! $\square$

**Lemma 2.12.** *Suppose $G(\mathbb{C})_i$ is a a simple normal subgroup of $G(\mathbb{C})$ and $\sigma$ is a righteous automorphism. Set $H_i := \pi_i(H)$, where $\pi_i$ is the projection on $G(\mathbb{C})_i$. If $H_i \neq G(\mathbb{C})_i$, then $\sigma(G(\mathbb{C})_i) = G(\mathbb{C})_i$.*

*Proof.* There exists an irreducible representation $R$ of $G(\mathbb{C})_i$ such that $V(R)^{H_i} \neq 0$. The rest of the proof runs as in the previous lemma. $\square$

Note that if the conditions of these lemmas are satisfied then the restriction of any righteous automorphism of $G(\mathbb{C})$ on $G(\mathbb{C})_i$ is a righteous automorphism of the spherical pair $(G(\mathbb{C})_i, H_i)$.

According to the classification there is only three indecomposable spherical pairs $(G(\mathbb{C}), H)$ such that $H_i = G(\mathbb{C})_i$ for some $i$. They are:

1) the symmetric pair of the form $(H \times H, H)$;

2) $(\mathrm{Sp}_2 \times \mathrm{Sp}_4 \times \mathrm{Sp}_2, \mathrm{Sp}_2 \times \mathrm{Sp}_2)$;

3) $(\mathrm{Sp}_{2n} \times \mathrm{Sp}_2 \times \mathrm{Sp}_2, \mathrm{Sp}_{2n-2} \times \mathrm{Sp}_2)$.

Let $\sigma$ be an automorphism of the second or third pair $(G(\mathbb{C}), H)$. Assume that $\sigma$ permutes simple factors of $G(\mathbb{C})$. It is easy to check that there are irreducible representations of $G(\mathbb{C})$

containing nonzero $H$-invariant vectors such that condition (c) is not satisfied. Let us indicate the highest weights of these representations. In the second case it is $\varpi_1(1)+\varpi_1(2)$; and in the third one they are $\varpi_1(1)+\varpi_1(2)$ and $\varpi_1(1)+\varpi_1(3)$ ($\varpi_i(n)$ being the $i$-th fundamental weight of the $n$-th simple factor of $G(\mathbb{C})$). For more details concerning this notation see [35, §2 of Capter 4 and Table 1 in the Reference Chapter].

**Proposition 2.13.** *Let $(G(\mathbb{C}) = G(\mathbb{C})_1 \times G(\mathbb{C})_2,\ H = H_1 \times H_2)$ be decomposable spherical pair. Suppose $\sigma_1$ and $\sigma_2$ are automorphisms of $G(\mathbb{C})_1$ and $G(\mathbb{C})_2$ respectively. The automorphism $\sigma = \sigma_1 \times \sigma_2$ is righteous if and only if $\sigma_1$ and $\sigma_2$ are righteous.*

*Proof.* We have $\mathfrak{g}(\mathbb{C})/\mathfrak{h} = \mathfrak{g}(\mathbb{C})_1/\mathfrak{h}_1 \oplus \mathfrak{g}(\mathbb{C})_2/\mathfrak{h}_2$. Moreover, $G(\mathbb{C})_i$ acts trivially on $\mathfrak{g}(\mathbb{C})_j$ for $i \neq j$ Hence, condition $(\mathrm{b})_{\mathbb{C}}$ holds for $\sigma$ if and only if it holds for $\sigma_1$ and $\sigma_2$. □

There exists a $G(\mathbb{C})$-invariant scalar product on $\mathfrak{g}(\mathbb{C})$. The space $\mathfrak{g}(\mathbb{C})/\mathfrak{h}$ is identified with the orthogonal complement $\mathfrak{h}^{\perp}$ to $\mathfrak{h}$ in $\mathfrak{g}(\mathbb{C})$.

Given a symmetric pair $(G(\mathbb{C}), H)$, denote by $\varphi$ the involution of $G(\mathbb{C})$ such that $H = (G(\mathbb{C})^{\varphi})^0$. Note that $\varphi$ is righteous. Let $\theta$ be the Weyl involution from Theorem 2.8 and *id* the identity mapping of $G$.

Krämer's paper [25] contains the decomposition of $\mathbb{C}[Y]$ into direct sum of irreducible $G(\mathbb{C})$-modules for all simple groups $G(\mathbb{C})$. This immediately shows that, in case of simple group $G(\mathbb{C})$, almost all automorphisms satisfying condition (c) are equivalent to a Weyl involution. The exceptions are listed in Theorem 2.14.

**Theorem 2.14.** *All indecomposable principal spherical pairs such that their righteous automorphisms are not equivalent to a Weyl involution are listed, up to local isomorphism, in Table 2.5. All righteous automorphisms of each pair are listed up to equivalence.*

Table 2.5.

| | $(G(\mathbb{C}), H)$ | $\sigma \in \mathrm{Aut}(G(\mathbb{C}), H)$ |
|---|---|---|
| 1 | $(\mathrm{SL}_{p+q},\ \mathrm{S}(\mathrm{GL}_p \times \mathrm{GL}_q))$, $(E_6,\ \mathrm{Spin}_{10} \cdot \mathbb{C}^*)$, $(E_6,\ \mathrm{SL}_6 \times \mathrm{SL}_2)$, $(\mathrm{SO}_{4n+2},\ GL_{2n+1})$ | $\mathrm{id}, \theta$ |
| 2 | $(\mathrm{SO}_{2(p+q)},\ \mathrm{SO}_{2p} \times \mathrm{SO}_{2q})$ $(p > q)$ | $\mathrm{id}, a(I_1)$ |
| 3 | $(\mathrm{SO}_{2(p+q)},\ \mathrm{SO}_{2p+1} \times \mathrm{SO}_{2q-1})$ $(p \geq q)$ | $\mathrm{id}, \varphi$ |
| 4 | $(H \times H,\ H)$ | $\theta, \varphi$ |

Here $I_1 = \mathrm{diag}(-1, 1, ...1)$.

*Proof.* Consider first the pairs with simple $G(\mathbb{C})$. For all pairs that are not contained in Table 2.5 condition (c) holds only for one connected component of $\mathrm{Aut}G(\mathbb{C})$, more precisely, for the connected component containing a Weyl involution. For the pairs contained in Table 2.5, condition (c) holds for two connected components of $\mathrm{Aut}\mathfrak{g}(\mathbb{C})$ (see [25]). Any automorphism indicated in Table 2.5 normalises $H$. Hence all of them are righteous by Theorem 2.10

For a symmetric space $(H \times H)/H$ there are nonzero $H$-invariant vectors in each irreducible representation of $H \times H$ of the form $R(\varpi_i(1) + \varpi_i(2)^*)$. Here $\{\varpi_i\}$ are fundamental weights of $H$. Suppose an automorphism $\Psi$ preserve the diagonal of $H \times H$ and satisfied condition (c). There are only two possibilities:
1) $\Psi(p, q) = (\psi(p), \psi(q))$; then $R(\varpi_i)^\psi = R(\varpi_i)^*$ for each $i$, hence $\Psi$ is equivalent to $\theta$;
2) $\Psi(p, q) = (\psi(q), \psi(p))$; then $R(\varpi_i)^\psi = R(\varpi_i)$ for each $i$, hence $\Psi$ is equivalent to $\varphi$.

Now we prove that for the first eight pairs of Table 2.3 all righteous automorphisms are equivalent to a Weyl involution. As we already know, a righteous automorphism does not permute simple direct factors $G(\mathbb{C})_i$ of $G(\mathbb{C})$. Recall that $H_i \subset G(\mathbb{C})_i$ is the image of the projection of $H$ on $G(\mathbb{C})_i$. Each righteous automorphism is of the form $\sigma = \sigma_1 \times \sigma_2 \times \sigma_3$ (or $\sigma_1 \times \sigma_2$), where $\sigma_i$ is a righteous automorphism of $(G(\mathbb{C})_i, H_i)$.

In cases 3, 4, 6, and 8 each simple direct factor of $G(\mathbb{C})$ has no outer automorphisms. In case 5, $\sigma_1$ has to be a Weyl involution and the second simple direct factor has no outer automorphisms.

In cases 1 and 7, we point out some irreducible representations of $G(\mathbb{C})$ containing nonzero $H$-invariant vectors. Using these representations one can easily verify that condition (c) holds for a single connected component of $\mathrm{Aut}\mathfrak{g}(\mathbb{C})$.

| Case | Representations |
|---|---|
| 1) | $\varpi_1^n(1) \cdot \varpi_n^{n+1}(2)$ |
| 7) $n = 7$ | $\varpi_4(1) \cdot \varpi_3(2), \varpi_3(1) \cdot \pi_3(2)$ |
| $n = 8$ | $\varpi_4(1) \cdot \varpi_3(2), \varpi_4(1) \cdot \pi_4(2)$ |
| $n \neq 7, 8$ | $\varpi_k(1) \cdot \varpi_p(2)$  $k = [(n+1)/2], p = [n/2]$ |

In case 2, the identity mapping is not righteous, because there is a homogeneous polynomial $f \in \mathbb{C}[\mathfrak{g}(\mathbb{C})/\mathfrak{h}]^H$ such that $\deg f$ is odd. Consider an $H$-invariant space

$$V \subset \mathfrak{g}(\mathbb{C})/\mathfrak{h}, V = \mathfrak{sl}_2 \oplus L_{2,n} \oplus L_{n,2} \subset \mathfrak{sl}_{n+2},$$

where $L_{p,q}$ is the space of complex $p \times q$-matrices. Take $\xi = (D, A, B) \in V$ ($D \in \mathfrak{sl}_2, A \in L_{2,n}, B \in L_{n,2}$), $h = (u, v, w, \lambda) \in H$ ( $u \in \mathrm{SL}_2, v \in \mathrm{SL}_n, w \in \mathrm{Sp}_{2m-2}, \lambda \in \mathbb{C}^*$). Then $h\xi = (uDu^{-1}, \lambda^{n+2}uAv^{-1}, \lambda^{-n-2}vBu^{-1})$. Suppose a vector $\eta$ lies in the $H$-invariant complement to $V$. Set $f(\xi + \eta) = f(\xi) = tr(DAB)$. Then $f \in \mathbb{C}[\mathfrak{g}(\mathbb{C})/\mathfrak{h}]^H$ and $\deg f = 3$. $\square$

Now we describe the righteous automorphisms of non-principal pairs up to equivalence. Let $(G(\mathbb{C}), H)$ be a non-principal indecomposable spherical pair. Then $\mathfrak{g}(\mathbb{C}) = \bigoplus \mathfrak{g}_i(\mathbb{C})$, $\mathfrak{h} = (\bigoplus \mathfrak{h}_i') \oplus \mathfrak{z}(\mathfrak{h})$, where $(\mathfrak{g}(\mathbb{C})_i, \mathfrak{h}_i)$ are indecomposable principal spherical pairs. For each $i$, we have $\dim \mathfrak{z}_i = 1$ and $\mathfrak{z}(\mathfrak{h}) \subset \bigoplus \mathfrak{z}_i$. (Here $\mathfrak{h}_j'$ denotes the derived algebra of $\mathfrak{h}_i$ and $\mathfrak{z}_i$ the centre of $\mathfrak{h}_i$.)

It is already proved that the righteous automorphisms preserve normal subgroups. Since $(G(\mathbb{C}), H)$ is indecomposable, for every $i$ there is an element $\xi \in \mathfrak{h}^\perp \cap \mathfrak{n}(\mathfrak{h})$ such that its projection to $\mathfrak{g}_i$ is non-zero. Note that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}' \oplus (\bigoplus \mathfrak{z}_i)$. The orbit $H\xi = \{\xi\}$ is closed. Thus in order to determine a righteous automorphism of $\mathfrak{g}(\mathbb{C})$, it is necessary and sufficient to indicate in each connected component of $\mathrm{Aut}\mathfrak{g}(\mathbb{C})$ a righteous automorphism $\sigma_i$ multiplying all vectors in $\mathfrak{z}_i$ by $-1$. (Of course, if such automorphism exists.)

**Proposition 2.15.** *Up to equivalence, all righteous automorphisms of a non-principal indecomposable spherical pair $(G(\mathbb{C}), H)$ are of the following form: $\sigma = \sigma_1 \times \dots \times \sigma_n$, where*

$$\sigma_i = \begin{cases} \theta_i, \ \text{if } (\mathfrak{g}(\mathbb{C})_i, \mathfrak{h}_i) \neq (\mathfrak{so}_{2n+2}, \mathfrak{so}_2 \oplus \mathfrak{so}_{2n}), (\mathfrak{sl}_{2n}, \mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)); \\ a(I_1) \ \text{or } a(I_{1,2n+2}), \ \text{if } (\mathfrak{g}(\mathbb{C})_i, \mathfrak{h}_i) = (\mathfrak{so}_{2n+2}, \mathfrak{so}_2 \oplus \mathfrak{so}_{2n}); \\ a(S_n) \ \text{or } \theta_i, \ \text{if } (\mathfrak{g}(\mathbb{C})_i, \mathfrak{h}_i) = (\mathfrak{sl}_{2n}, \mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_n)). \end{cases}$$

*Here* $I_1 = \mathrm{diag}(-1, 1, \dots 1)$, $I_{1,2n+2} = \mathrm{diag}(-1, 1, \dots 1, -1)$, $S_n = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}$.

*Proof.* If the pair $(\mathfrak{g}(\mathbb{C})_i, \mathfrak{h}_i)$ does not belong to the following list

$$(\mathfrak{sl}_{p+q}, \mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q)), (E_6, \mathfrak{so}_{10} \oplus \mathbb{C}), (\mathfrak{so}_{2n+2}, \mathfrak{so}_{2n} \oplus \mathfrak{so}_2),$$

then, by Theorem 2.14, $\sigma_i = \theta_i$. Each of these three pairs has two non-equivalent righteous automorphisms. The pairs $(E_6, \mathfrak{so}_{10} \oplus \mathbb{C})$ and $(\mathfrak{sl}_{p+q}, \mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q))$, with $p \neq q$, have no inner automorphisms multiplying all vectors in the centre of $\mathfrak{h}$ by $-1$. Besides all there outer automorphisms are equivalent to Weyl involutions. □

### 2.2.3 The real case

We assume that $G(\mathbb{C})$ is simply connected. Let $(G = G_1 \times G_2, K = K_1 \times K_2)$ be a decomposable weakly symmetric pair. Then its complexification $(G(\mathbb{C}) = G_1(\mathbb{C}) \times G_2(\mathbb{C}), H = H_1 \times H_2)$ is a decomposable spherical pair. As we already know, all righteous automorphisms of $(G(\mathbb{C}), H)$ preserve $G_1(\mathbb{C})$ and $G_2(\mathbb{C})$. Hence, each righteous automorphism $\sigma$ of $G$ is the product of automorphisms $\sigma_1$ and $\sigma_2$ of $G_1$ and $G_2$, respectively. Similarly to the complex case we have

**Proposition 2.16.** *The automorphism $\sigma = \sigma_1 \times \sigma_2$ is righteous if and only if $\sigma_1$ and $\sigma_2$ are righteous.*

Thus it suffices to describe the righteous automorphisms of indecomposable weakly symmetric spaces. We have proved in the first section of this chapter, that each indecomposable weakly symmetric space is a real form of an indecomposable spherical space. Besides, we have described non-compact weakly symmetric homogeneous spaces, Theorem 2.2.

**Lemma 2.17.** *Let $\sigma \in \mathrm{Aut}G(\mathbb{C})$ be a righteous automorphism of a complex indecomposable spherical pair $(G(\mathbb{C}), H)$. Then in the notation of Theorem 2.2 we have $\sigma\varphi = \varphi\sigma$.*

*Proof.* It suffices to prove that $\sigma(G^\varphi) = G^\varphi$. Observe that both these groups contain $H$. For symmetric pairs our statement is tautological. For all pairs with a simple group $G(\mathbb{C})$, except $(\mathrm{Spin}_8, G_2)$, there is only one subgroup of the form $G(\mathbb{C})^\varphi$ containing $H$, [46, Lemmas 2,3].

For $(\mathrm{Spin}_8, G_2)$, the automorphism $\sigma$ is inner. Hence the involutions $\varphi$ and $\sigma\varphi\sigma^{-1}$ lies in the same connected component of $\mathrm{Aut}(\mathrm{Spin}_8)$. This means that both $\varphi$ and $\sigma\varphi\sigma^{-1}$ keep intact some non-trivial central element of $\mathrm{Spin}_8$. Assume that $\varphi \neq \sigma\varphi\sigma^{-1}$. Then the group $\mathrm{SO}_8 = \mathrm{Spin}_8/\mathbb{Z}_2$ has two involution acting trivially on the spherical subgroup $G_2$. But this is not true according to [46, Theorem 3].

Suppose $G(\mathbb{C}) = G_1(\mathbb{C}) \times ... \times G_m(\mathbb{C})$, where $G_i(\mathbb{C})$ are simple groups. As we already know, $\sigma$ does not permute the simple direct factors of non-symmetric pairs. If $\sigma$ and $\varphi$ preserve some $G_i(\mathbb{C})$, then their restrictions to $G_i(\mathbb{C})$ commute. On the other hand, if the involution $\varphi$ permutes two components $G_i(\mathbb{C})$ and $G_j(\mathbb{C})$ (which is only possible for $(\mathrm{Sp}_2 \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2k}, \mathrm{Sp}_2 \times \mathrm{Sp}_{2k-2})$), then the diagonal of $G_i(\mathbb{C}) \times G_j(\mathbb{C})$ is contained in the projection of $H$ to $G_i(\mathbb{C}) \times G_j(\mathbb{C})$, that is, $\sigma$ preserves the set of $\varphi$-invariant elements. $\square$

**Corollary.** *Let $(G, K)$ and $(G_0, K)$ be weakly symmetric pairs corresponding to a spherical pair $(G(\mathbb{C}), H)$. Then the sets of the righteous automorphisms of $G$ and $G_0$, regarded as subsets of $\mathrm{Aut}(G(\mathbb{C}), H)$, coincide.*

*Proof.* Theorem 2.8 yields a bijection between the set of the righteous automorphisms of $G(G_0)$ and the set of the righteous automorphisms of $(G(\mathbb{C}), H)$ preserving the real form $G(G_0)$. We can assume that $G = G(\mathbb{C})^\tau$, where $\tau$ is a compact real structure on $G(\mathbb{C})$, and $G_0 = G(\mathbb{C})^{\varphi\tau}$. If $\sigma$ is a righteous automorphism of $(G(\mathbb{C}), H)$, then

$$\sigma(G_0) = G_0 \Longleftrightarrow \sigma\tau\varphi = \tau\varphi\sigma \Longleftrightarrow \sigma\tau = \tau\sigma \Longleftrightarrow \sigma(G) = G.$$

$\square$

Let $\sigma$ be a righteous automorphism of $(G, K)$. Then $a(g)\sigma$ $(g \in G(\mathbb{C}))$ is a righteous automorphism of $(G, K)$ if and only if $g \in N(K) \cap N(G)$. If $N(G) \neq G$, then some equivalent righteous automorphisms of $(G(\mathbb{C}), H)$, preserving $G$, can be non-equivalent as righteous automorphisms of $(G, K)$. But for a compact real form, we have $N(G) = G$. Thus, righteous automorphisms of $(G, K)$ are equivalent as automorphisms $G(\mathbb{C})$ if and only if they are equivalent as automorphisms of $G$.

Assume that $G$ is compact. For each semisimple automorphism $\psi$ of $G(\mathbb{C})$, there is a real structure $\tau$ commuting with $\psi$. Each righteous automorphism of $G(\mathbb{C})$ is equivalent to either an involution or the identity mapping. Hence, we can assume that the automorphisms listed in the previous subsection are righteous automorphism of corresponding compact groups. All righteous automorphisms can be obtained from them by multiplication by the elements of $\mathrm{Int}(G, K)$. In particular, each righteous automorphism of $G$ is of the form $\sigma = a(n)\delta$, where $n \in N(K)$ and $\delta$ is an involutive righteous automorphism. We have

$$\sigma^2 = a(n\delta(n)) = a(n(n^{-1}k)) = a(k), \text{ where } k \in K,$$

i.e., the diffeomorphism $s^2$ corresponding to $\sigma^2$, acts on $X$ as $k$. Hence, if we consider $G$ as a subgroup of $\mathrm{Diff}(M)$, then $s^2 \in G$ for each righteous diffeomorphism $s$. Thus, for semisimple groups our definition of a weakly symmetric space is equivalent to the original definition given by Selberg.

### 2.2.4 Non-symmetric weakly symmetric Riemannian manifolds

Let $M$ be a connected Riemannian manifold and $\mathrm{Isom}(M)$ be the full isometry group of $M$.

**Definition 13.** The manifold $M$ is said to be *symmetric*, if for every point $x \in M$ there is an isometry $s \in \mathrm{Isom}(M)$ such that $s(x) = x$ and $d_x s = -\mathrm{id}$.

The notion of a weakly symmetric manifold is a generalisation of a notion of a symmetric manifold.

**Definition 14.** The manifold $M$ is said to be *weakly symmetric*, if the following equivalent conditions hold:
(e) $\forall x, y \in M \;\; \exists s \in \mathrm{Isom}(M) : s(x) = y, s(y) = x$.
(f) $\forall x \in M \;\; \forall \xi \in T_x(M) \;\; \exists s \in \mathrm{Isom}(M) : s(x) = x, \;\; ds_x(\xi) = -\xi$.
(The equivalence of the given conditions is proved, for example, in [6].)

The geometric meaning of Definition 14 is that for every point $x \in M$ and for every geodesic line containing $x$ there is an isometry stabilising $x$ and reversing the geodesic line.

Let $G$ be a real Lie group and $K \subset G$ a compact subgroup. Assume that $M = G/K$ is connected.

**Definition 14′.** The homogeneous space $M$ is called *symmetric*, if there is an involution $\sigma$ of $G$ such that $(G^\sigma)^0 \subset K \subset G^\sigma$.

Let $M$ be a weakly symmetric Riemannian manifold. Then, due to condition (e), $\mathrm{Isom}(M)$ acts on $M$ transitively, in particular, $M$ is a complete Riemannian manifold. Denote by $\mathrm{St}(x) \subset \mathrm{Isom}(M)$ the stabiliser of a point $x \in M$. We have

$$M = \mathrm{Isom}(M)/\mathrm{St}(x) = \mathrm{Isom}(M)^0/(\mathrm{St}(x) \cap \mathrm{Isom}(M)^0),$$

$\mathrm{Isom}(M)$ is a real Lie group, $\mathrm{St}(x)$ is a compact subgroup of $\mathrm{Isom}(M)$. It is easy to verify, that the homogeneous space $M$ of the group $\mathrm{Isom}(M)$ is weakly symmetric with respect to the identity map. The analogous fact concerning symmetric manifolds is well known. A symmetric Riemannian manifold $M$ is a symmetric homogeneous space of the group $\mathrm{Isom}(M)$ and also of the group $\mathrm{Isom}(M)^0$.

On the other hand, suppose that a homogeneous space $M = G/K$ is (weakly) symmetric with respect to some automorphism $\sigma$. Recall that $\sigma$ defines an automorphism $s$ of $M$ by the formula $s(gK) = \sigma(g)K$ and this $s$ satisfies condition (2.2). Let us introduce a $G$-invariant Riemannian metric on $M$. Then $s \in \mathrm{Isom}(M)$ and $G/K$ becomes a (weakly) symmetric Riemannian manifold.

Thus, the notions of (weakly) symmetric Riemannian manifolds and (weakly) symmetric homogeneous space are quite close. The difference between them lays in the fact that a weakly symmetric Riemannian manifold $M$ can be a weakly symmetric homogeneous space of several groups.

The aim of this subsection is the classification of non-symmetric weakly symmetric Riemannian manifolds with reductive isometry group. Among weakly symmetric homogeneous spaces of reductive Lie groups, we distinguish non-symmetric Riemannian manifolds.

Let $M = G/K$ be a weakly symmetric homogeneous space. Then $M$ is also a weakly symmetric homogeneous space of $G^0$, see [43, Lemma 1]. From now on assume that $G$ is connected. This restriction is not important for our goal, i.e., classification of non-symmetric weakly symmetric Riemannian manifolds.

A non-symmetric homogeneous space can be a symmetric Riemannian manifold. For instance, a sphere $\mathrm{S}^{2n-1} = \mathrm{SU}_n/\mathrm{SU}_{n-1} = \mathrm{SO}_{2n}/\mathrm{SO}_{2n-1}$ is a symmetric Riemannian manifold and simultaneously a non-symmetric weakly symmetric homogeneous space of the group $\mathrm{SU}_n$.

**Definition 15.** Let $M = G/K$ be a homogeneous space, $\mu$ be a $G$-invariant Riemannian metric on $M$. We say that $\mu$ is *(weakly) symmetric*, if the pair $(M, \mu)$ is a (weakly) symmetric Riemannian manifold.

To understand whether a given Riemannian metric is symmetric or not, it is sufficient to know the isometry group of the pair $(M, \mu)$ or its connected component $\mathrm{Isom}(M)^0$.

Denote by $\mathfrak{m}$ the tangent space $T_{eK}(G/K) \cong \mathfrak{g}/\mathfrak{k}$. The space $\mathfrak{m}$ can be identified with a $K$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. Let $\mathbb{B}(\mathfrak{m})$ be a set of all positive-definite $K$-invariant scalar products on the vector space $\mathfrak{m}$. In order to determine a $G$-invariant Riemannian metric on $G/K$, it is necessary and sufficient to choose an element of $\mathbb{B}(\mathfrak{m})$.

As was proved in [43], a connected homogeneous space that is locally isomorphic to a weakly symmetric one is also weakly symmetric. Unfortunately, the analogous statement is not true for symmetric spaces.

Let $\widetilde{M} = \widetilde{G}/\hat{K}$ be a simply connected covering of $M$. Here $\widetilde{G}$ is a simply connected covering of $G$ and $\hat{K} \subset \widetilde{G}$ is a connected subgroup with $\mathrm{Lie}\,\hat{K} = \mathfrak{k}$. Any $G$-invariant

Riemannian metric $\mu$ on $M$ can be lifted to a $\widetilde{G}$-invariant Riemannian metric $\widetilde{\mu}$ on $\widetilde{M}$. Now we will show how to decide whether $M$ is symmetric, assuming that $\mathrm{Isom}(\widetilde{M})$ is known.

Suppose $(M, \mu)$ is a symmetric Riemannian manifold. Then $(\widetilde{M}, \widetilde{\mu})$ is also symmetric. Moreover, $M$ is a symmetric homogeneous space of $\mathrm{Isom}(\widetilde{M})^0$.

On the other hand, suppose we know that $(\widetilde{M}, \widetilde{\mu})$ is symmetric. Denote by $F$ the group $\mathrm{Isom}(\widetilde{M})$. If $F$ acts on $M$, then, by the principal result of [17], $M$ is a symmetric homogeneous space of $F/N$, where $N$ is the ineffective kernel of $F : M$. Thus, $M$ is symmetric if and only $F$ acts on $M$.

We have reduced the problem of finding all non-symmetric Riemannian manifold among the weakly symmetric homogeneous spaces to the same problem for simply connected homogeneous spaces.

We will deal with homogeneous spaces of reductive groups. But before we restrict ourself to this case, note that it is quite possible that $\widetilde{M}$ is not a homogeneous space of any reductive group even if $M$ is. For example, there is no reductive group acting transitively on a real line $\mathbb{R} = \widetilde{\mathrm{U}_1}$.

In particular, having restricted the area of our consideration to homogeneous spaces of reductive groups, we might loose some weakly symmetric spaces. To avoid this unhappy event, we prove the following lemma.

**Lemma 2.18.** *Let $M = G/K$ be a homogeneous space of a reductive group $G$ and $F := G'K \neq G$. Then $\widetilde{M}$ decomposes, as a Riemannian manifold, into a product $\widetilde{M} = M_s \times M_0$, where $M_s$ is a simply connected covering of the homogeneous space $G'/(G' \cap K)$ and $M_0$ is a locally euclidian symmetric manifold.*

*Proof.* Let $\mu$ be a $G$-invariant Riemannian metric on $M$, determined by an element $b \in \mathbb{B}(\mathfrak{m})$. Denote by $\mathfrak{m}_1$ an orthogonal complement of $\mathfrak{f}/\mathfrak{k}$ in $\mathfrak{m}$. Evidently, $\mathfrak{m}_1 \subset \mathfrak{z}(\mathfrak{g})$. There is an equality of Riemannian manifolds $\widetilde{M} = M_s \times M_0$, where $M_0$ is a locally euclidian symmetric Riemannian manifold and $M_s$ is a simply connected covering of $F/K$ endowing with a $F$-invariant Riemannian metric determined by a scalar product $b|_{\mathfrak{f}/\mathfrak{k}}$. To conclude the proof, note that because $F = G'K$, we have $F/K = G'/(G' \cap K)$. $\square$

**Corollary.** *A Riemannian manifold $\widetilde{M}$ is symmetric if and only if $M_s$ is symmetric; moreover, if $M_s$ is symmetric, then* $\mathrm{Isom}(M_s)$ *is semisimple and there is an equality* $\mathrm{Isom}(\widetilde{M}) = \mathrm{Isom}(M_s) \times \mathrm{Isom}(M_0)$.

*Proof.* The product of two Riemannian manifolds is symmetric if and only if each of them is symmetric, hence the first statement is true.

Suppose $M_s$ is symmetric. Assume that $\mathrm{Isom}(M_s)$ is not semisimple, i.e., $M_s$ is locally isomorphic to the product $M_1 \times \mathbb{R}^n$, where $M_1$ is a symmetric homogeneous space of a semisimple group $\mathrm{Isom}(M_1)$ and $n \geq 1$. According to the conditions of the lemma, the group $\widetilde{G'}$ acts on $M_s$ transitively. Hence, $\widetilde{G'}$ has a non-trivial connected centre. This leads to a contradiction, because $G'$ is semisimple. Since $M_s$ is a symmetric homogeneous space of a semisimple group, each isometry of $\tilde{M}$ preserves $M_s$ and $M_0$, see [20]. $\square$

From now on, we assume that $G$ is connected and reductive, $G/K$ is simply connected, and $G = G'K$.

**Theorem 2.19.** *[47, Theorem 1] Suppose* $(G, K) = (G_1, K_1) \times (G_2, K_2)$ *is a weakly symmetric pair, and set* $M_i = G_i/K_i$*. Then, regardless of the choice of a* $G$*-invariant symmetric metric on* $M = G/K$*, there is a decomposition* $M = M_1 \times M_2$ *in the sense of Riemannian manifolds.*

If $N_{G_1}(K_1)^0 = K_1$ or $N_{G_2}(K_2)^0 = K_2$, then it is true for any $G$-invariant metric. In the general case the statement is more complicated. As a corollary of Theorem 1 we have: each symmetric metric on $M$ is a product of symmetric metrics on $M_1$ and $M_2$. In particular, a symmetric metric on $M$ exists if and only if such metrics exist on both $M_1$ and $M_2$. Thus, in order to classify all weakly symmetric non-symmetric Riemannian manifolds, it suffices to consider only indecomposable weakly symmetric homogeneous spaces.

Denote by $Z(G)$ the centre of $G$. Set $K_r := K/(K \cap Z(G))$. A homogeneous space $G'/K_r$ is called the *central reduction* of $G/K$. Recall that $G/K$ is weakly symmetric if and only of the central reduction of $G/K$ is weakly symmetric.

Suppose a pair $(P, Q)$ of Lie groups is an extension of $(G, K)$. If $P$ is a symmetric subgroup of $Q$, we call $(P, Q)$ a *symmetric extension* of $(G, K)$.

Let $M = G/K$ be a homogeneous space. Suppose the pair $(G, K)$ is *effective*, i.e., $K$ contains no non-trivial normal subgroups of $G$. (Note that a pair $(\mathrm{Isom}(M), \mathrm{St}(x))$ is always effective.) For each $G$-invariant Riemannian metric $\mu$ on $M$, the pair $(\mathrm{Isom}(M), \mathrm{St}(x))$ is an extension of $(G, K)$, if of course $G \neq \mathrm{Isom}(M)$. In order to find all symmetric $G$-invariant Riemannian metrics on $M$ it is necessary and sufficient to describe all symmetric extensions of $(G, K)$. Or equivalently, to each symmetric pair find out all weakly symmetric pair which it extends. As was shown above, if $M$ is a symmetric manifold, then $\mathrm{Isom}(M)$ is semisimple.

*Remark* 2. It can be shown, that $\mathrm{Isom}(M)$ is always reductive. Moreover, if $M$ is a symmetric homogeneous space of $G$, then, due to results of Cartan and Helgason we have $G = \mathrm{Isom}(M)^0$.

If $N$ is a normal subgroup of $G$ and $N \subset K$, then $G/K = (G/N)/(K/N)$ and $N$ acts on $M$ trivially. Therefore, we consider only effective pairs. Recall that a $G$-invariant Riemannian metric on $M$ is determined by an element of the set $\mathbb{B}(\mathfrak{m})$, i.e., by a $K$-invariant scalar product on $\mathfrak{m}$. Note that the normaliser $N_G(K)$ naturally acts on $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$.

**Lemma 2.20.** *Let* $G/K$ *be a weakly symmetric homogeneous space. Then each* $K$*-invariant scalar product on* $\mathfrak{m}$ *is also* $N_G(K)$*-invariant.*

*Proof.* According to Corollary 2 of Lemma 2.6, the orbits of $K$ and $N(K)$ on $\mathfrak{m}$ coincide. Hence, these groups have the same invariants in $\mathbb{R}[\mathfrak{m}]$. In particular, $S^2(\mathfrak{m})^K = S^2(\mathfrak{m})^{N(K)}$. $\square$

**Corollary 1.** *The set* $\mathbb{B}(\mathfrak{m})$ *depends only on the pair* $(\mathfrak{g}, \mathfrak{k})$*, i.e., the set of weakly symmetric invariant Riemannian metrics on a simply connected homogeneous space* $G/K$ *coincides with the analogous set on any homogeneous space locally isomorphic to* $G/K$*.*

Suppose $(G, K)$ is a weakly symmetric pair such that $N(K)^0 \neq K$. Then $G/K = (N \times G)/N(K)$, where $N = N(K)/K$ acts on $G/K$ by right multiplications.

**Corollary 2.** *Each* $G$*-invariant Riemannian metric on* $G/K$ *is also* $(N \times G)$*-invariant.*

Let $M = G/K$ be a weakly symmetric homogeneous space. Suppose $G$ has a non-trivial connected centre, i.e., $G' \neq G$. A pair $(G', G' \cap K)$ is called a *truncated* weakly symmetric. Denote by $K$" a group $G' \cap K$. Note that a truncated weakly symmetric pair might be or might be not weakly symmetric. For example, a pair $(\mathrm{SU}_{n+1}, \mathrm{SU}_n)$ is a truncation of $(\mathrm{U}_{n+1}, \mathrm{U}_n)$ and a non-weakly symmetric pair $(\mathrm{SU}_{2n}, \mathrm{SU}_n \times \mathrm{SU}_n)$ is a truncation of a weakly symmetric pair $(\mathrm{U}_{2n}, \mathrm{U}_n \times \mathrm{SU}_n)$.

**Corollary 3.** *If* $(G', K")$ *is a weakly symmetric pair, then the sets of* $G$*- and* $G'$*-invariant Riemannian metrics on* $M$ *coincide.*

The last statement means that the homogeneous spaces $G/K$ and $G'/K$" correspond exactly to the one and the same weakly symmetric Riemannian manifold.

As proved in [1], $N$ is an Abelian group. In particular, replacing $G$ by $G \times N$ is, in a sense, a process inverse to a truncation.

**Lemma 2.21.** *Let* $(G, K)$ *be an indecomposable weakly symmetric pair, where* $\mathfrak{z}(\mathfrak{g}) \neq 0$*. Suppose the corresponding truncated pair is decomposable, i.e.,* $(G', G' \cap K) = (G_1, K_1) \times (G_2, K_2)$*. Then there are weakly symmetric pairs* $(\check{G}_1, \check{K}_1)$ *and* $(\check{G}_2, \check{K}_2)$ *such that their product is an extension of* $(G, K)$ *and the sets of* $G$*- and* $(\check{G}_1 \times \check{G}_2)$*-invariant Riemannian metrics on* $G/K$ *coincide.*

*Proof.* Set $(\check{G}_1, \check{K}_2) := (G/G_2, K/(K \cap G_2))$ and $(\check{G}_2, \check{K}_2) := (G/G_1, K/(K \cap G_1))$. Note that there is a decomposition $\check{G}_1 \times \check{G}_2 = G'(\check{K}_1 \times \check{K}_2)$. The group $G$ is embedded in a natural way into $\check{G}_1 \times \check{G}_2$, here the centre of $G$ is embedded diagonally into the product of centres of $G/G_2$ and $G/G_1$. Evidently, $(G/G_2) \times (G/G_1) = G(\check{K}_1 \times \check{K}_2)$, hence, the product $(\check{G}_1, \check{K}_2) \times (\check{G}_2, \check{K}_2)$ is really an extension of $(G, K)$.

The pair $(G/G_2 \times G/G_1, K)$ is weakly symmetric, because its central reduction coincides with the central reduction of $(G, K)$. Evidently, $K/(K \cap G_2) \times K/(K \cap G_1) \subset N(K)$ (here we consider the normaliser in $\check{G}_1 \times \check{G}_2$). In particular, the orbits of $K$ and $\check{K}_1 \times \check{K}_2$ in $\mathfrak{m} = \mathfrak{g}_1/\mathfrak{k}_1 \oplus \mathfrak{g}_1/\mathfrak{k}_2$ are the same. To conclude the proof, note that $T_{eK}(G/K)$ is isomorphic to $\mathfrak{m}$ as a $K$-, $N(K)$- and, hence, $\check{K}_1 \times \check{K}_2$-module. □

**Corollary.** *Every symmetric metric on* $G/K$ *is a product of symmetric metrics on* $G_1/K_1$ *and* $G_2/K_2$*.*

In what follows we will consider only those weakly symmetric spaces $G/K$, whose truncated pairs $(G', K")$ are indecomposable. As was already shown, $(G', N_{G'}(K"))$ is a weakly symmetric pair, moreover, it is indecomposable if $(G', K")$ is indecomposable. In particular, we can assume that the list of the indecomposable truncated pairs is known.

Recall that a weakly symmetric space $G/K$ under consideration is also a homogeneous space of semisimple group $G'$.

**Theorem 2.22.** *An indecomposable (as a homogeneous space) non-symmetric non-compact homogeneous space of a semisimple group $G$ is not a symmetric Riemannian manifold regardless of the choice of a $G$-invariant metric.*

*Proof.* Assume that $M = G/K$ is symmetric. Decompose it into a product of indecomposable Riemannian manifolds. Let $M_n$ be a product of all non-compact factors, i.e., a symmetric space of negative curvature. Suppose a semisimple group $H \subset \mathrm{Isom}(M_n)$ acts transitively on $M_n$. By the Karpelevich theorem, see [23, Theorem 1], there is a Cartan involution $\sigma$ of $\mathrm{Isom}(M_n)$ such that $\sigma(H) = H$ and $\sigma|_{\mathfrak{h}}$ is a Cartan involution of $\mathfrak{h}$. In particular, $\mathfrak{h} = \mathfrak{k}_1 \oplus \mathfrak{m}_1$, $\mathfrak{m}_1 \subset \mathfrak{m}$. But then $\mathfrak{m}_1 = \mathfrak{m}$. Because $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m} = \mathfrak{isom}(M_n)$, we have $\mathfrak{h} = \mathfrak{isom}(M_n)$. Hence, any connected semisimple subgroup of $\mathrm{Isom}(M_n)$ acting transitively on $M_n$ coincides with $\mathrm{Isom}(M_n)^0$. The group $\mathrm{Isom}(M_n)$ can not act non-trivially on a compact or locally euclidian symmetric Riemannian manifold. This means that $G$ contains $\mathrm{Isom}(M_n)^0$ as a factor. Thus, if $M = M_n$, then the homogeneous space $G/K$ is symmetric and if $M \neq M_n$, then $G/K$ is reducible (as a homogeneous space). □

In Table 2.6 we present the principal result of [47]. We give the list of all simply connected compact indecomposable weakly symmetric homogeneous spaces of reductive Lie groups whose central reductions are principal indecomposable homogeneous spaces. If a homogeneous space $M$ in Table 2.6 admits a $G$-invariant symmetric metric, then we also indicate the identity component $P$ of the isometry group, and the stabiliser $Q$ in $P$ of a point in $M$. The pair $(P, Q)$ is a symmetric extension of $(G, K)$. The dimensions of the cones of $K$-invariant and $Q$-invariant (in brackets) positive-definite scalar products on the space $\mathfrak{m} = \mathfrak{g}/\mathfrak{k}$, i.e., the dimensions of the sets of weakly symmetric and symmetric metrics, are given in the fifth column of Table 2.6.

Suppose $G$ has a non-trivial connected centre and $G/K$ is also a weakly symmetric homogeneous space of $G'$. Clearly $G/K = G'/(G' \cap K)$. In Table 2.6 we list only one of these two homogeneous spaces, usually the former. In rows 5, 7, 17, 23b and 24 the group $\mathrm{Isom}(G/K)^0$ is equal not to $P$ but to its quotient by some central subgroup.

It can be seen that for almost all homogeneous spaces listed in Table 2.6 there are $G$-invariant metrics on $M$, which are not $P$-invariant for any $P$. Thus, there are non-symmetric weakly symmetric Riemannian metrics on these homogeneous spaces. Cases 14 and 16 are the only exceptions.

Table 2.6.

| | $M=G/K$ | $P{=}Isom(M)^0$ | $Q$ | $\dim\mathbb{B}(\mathfrak{m})$ |
|---|---|---|---|---|
| 1a | $\mathrm{SU}_n/\mathrm{SU}_{n-1}$ | $\mathrm{SO}_{2n}$ | $\mathrm{SO}_{2n-1}$ | 2(1) |
| 1b | $\mathrm{SU}_n/(\mathrm{SU}_{n-k}{\times}\mathrm{SU}_k)\ \ (n\neq n-k)$ | | | |
| 1c | $\mathrm{U}_{2n}/(\mathrm{U}_n{\times}\mathrm{SU}_n)$ | | | |
| 2 | $\mathrm{SU}_{2n+1}/(\mathrm{Sp}_n{\times}\mathrm{U}_1)$ | | | |
| 3 | $\mathrm{SU}_{2n+1}/\mathrm{Sp}_n$ | $\mathrm{SU}_{2n+2}$ | $\mathrm{Sp}_{n+1}$ | 3(1) |
| 4 | $(\mathrm{U}_1{\times}\mathrm{Sp}_n)/\mathrm{U}_n$ | | | |
| 5 | $\mathrm{Sp}_n/(\mathrm{Sp}_{n-1}{\times}\mathrm{U}_1)$ | $\mathrm{SU}_{2n}$ | $\mathrm{U}_{2n-1}$ | 2(1) |
| 6 | $(\mathrm{U}_1{\times}\mathrm{Sp}_n)/(\mathrm{Sp}_{n-1}{\times}\mathrm{U}_1)$ | | | |
| 7 | $\mathrm{SO}_{2n+1}/\mathrm{U}_n$ | $\mathrm{SO}_{2n+2}$ | $\mathrm{U}_{n+1}$ | 2(1) |
| 8 | $(\mathrm{U}_1{\times}\mathrm{SO}_{2n+1})/\mathrm{U}_n$ | | | |
| 9 | $(\mathrm{U}_1{\times}\mathrm{SO}_{4n+2})/\mathrm{U}_{2n+1}$ | | | |
| 10 | $\mathrm{SO}_{10}/(\mathrm{Spin}_7{\times}\mathrm{SO}_2)$ | | | |
| 11 | $(\mathrm{U}_1{\times}\mathrm{SO}_{10})/(\mathrm{Spin}_7{\times}\mathrm{SO}_2)$ | | | |
| 12 | $\mathrm{SO}_9/\mathrm{Spin}_7$ | $\mathrm{SO}_{16}$ | $\mathrm{SO}_{15}$ | 2(1) |
| 13 | $\mathrm{Spin}_8/G_2$ | $\mathrm{SO}_8\times\mathrm{SO}_8$ | $\mathrm{SO}_7\times\mathrm{SO}_7$ | 3(2) |
| 14 | $\mathrm{Spin}_7/G_2$ | $\mathrm{SO}_8$ | $\mathrm{SO}_7$ | 1(1) |
| 15 | $E_6/\mathrm{Spin}_{10}$ | | | |
| 16 | $G_2/\mathrm{SU}_3$ | $\mathrm{SO}_7$ | $\mathrm{SO}_6$ | 1(1) |
| 17 | $(\mathrm{U}_{n+1}\times\mathrm{SU}_n)/\mathrm{U}_n$ | $\mathrm{SU}_{n+1}{\times}\mathrm{SU}_{n+1}$ | $\mathrm{SU}_{n+1}$ | 2(1) |
| 18 | $(\mathrm{SU}_n{\times}\mathrm{Sp}_m)/(\mathrm{U}_{n-2}{\times}\mathrm{SU}_2{\times}\mathrm{Sp}_{m-1})$ | | | |
| 19a | $(\mathrm{Sp}_n\times\mathrm{Sp}_l\times\mathrm{Sp}_m)/$<br>$(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_{l-1}\times\mathrm{Sp}_{m-1})$ | | | |
| 19b | $(\mathrm{Sp}_n\times\mathrm{Sp}_1\times\mathrm{Sp}_m)/$<br>$(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_{m-1})$ | $\mathrm{SO}_{4n}\times\mathrm{SO}_{4m}$ | $\mathrm{SO}_{4n-1}{\times}\mathrm{SO}_{4m-1}$ | 6(2) |
| 19b′ | $(\mathrm{Sp}_1\times\mathrm{Sp}_n\times\mathrm{Sp}_1)/$<br>$(\mathrm{Sp}_1\times\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1)$ | $\mathrm{SO}_{4n}\times\mathrm{SO}_{4n}$ | $\mathrm{SO}_{4n-1}{\times}\mathrm{SO}_{4n-1}$ | 5(2) |
| 20 | $(\mathrm{Sp}_n\times\mathrm{SU}_4)/(\mathrm{Sp}_{n-2}\times\mathrm{SU}_4)$ | | | |
| 21 | $(\mathrm{SU}_n{\times}\mathrm{Sp}_m)/(\mathrm{SU}_{n-2}{\times}\mathrm{SU}_2{\times}\mathrm{Sp}_{m-1})$ | | | |
| 22a | $(\mathrm{Sp}_n\times\mathrm{Sp}_2\times\mathrm{Sp}_m)/$<br>$(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1\times\mathrm{Sp}_1\times\mathrm{Sp}_{m-1})$ | | | |
| 22b | $(\mathrm{Sp}_1\times\mathrm{Sp}_2\times\mathrm{Sp}_1)/(\mathrm{Sp}_1\times\mathrm{Sp}_1)$ | $\mathrm{Sp}_2\times\mathrm{Sp}_2$ | $\mathrm{Sp}_2$ | 3(1) |
| 23 | $(\mathrm{SO}_{n+1}\times\mathrm{SO}_n)/\mathrm{SO}_n$ | $\mathrm{SO}_{n+1}{\times}\mathrm{SO}_{n+1}$ | $\mathrm{SO}_{n+1}$ | 2(1) |
| 24a | $(\mathrm{Sp}_n{\times}\mathrm{Sp}_m)/(\mathrm{Sp}_{n-1}{\times}\mathrm{Sp}_1{\times}\mathrm{Sp}_{m-1})$ | | | |
| 24b | $(\mathrm{Sp}_n\times\mathrm{Sp}_1)/(\mathrm{Sp}_{n-1}\times\mathrm{Sp}_1)$ | $\mathrm{SO}_{4n}$ | $\mathrm{SO}_{4n-1}$ | 3(1) |

# Chapter 3

# Commutative homogeneous spaces of Heisenberg type

In this chapter, we consider homogeneous spaces of the form $(N \rtimes K)/K$. Here $S(\mathfrak{g}/\mathfrak{k})^K = S(\mathfrak{n})^K$. Therefore,we may assume that $\mathfrak{n}$ is a non-Abelian Lie algebra.

Decompose $\mathfrak{n}/\mathfrak{n}'$ into a sum of irreducible $K$-invariant subspaces, namely $\mathfrak{n}/\mathfrak{n}' = \mathfrak{w}_1 \oplus \ldots \oplus \mathfrak{w}_p$. According to [43, Prop. 15], if $X$ is commutative, then $[\mathfrak{w}_i, \mathfrak{w}_j] = 0$ for $i \neq j$, also $[\mathfrak{w}_i, \mathfrak{w}_i] = 0$ if there is $j \neq i$ such that $\mathfrak{w}_i \cong \mathfrak{w}_j$ as a $K$-module. Denote by $\mathfrak{n}_i := \mathfrak{w}_i \oplus [\mathfrak{w}_i, \mathfrak{w}_i]$ the subalgebra generated by $\mathfrak{w}_i$. Let $\mathfrak{v}^i$ be a $K$-invariant complement of $\mathfrak{n}_i$ in $\mathfrak{n}$. Denote by $K^i$ be the identity component of $K_*(\mathfrak{v}^i)$.

**Theorem 3.1.** *([48, Theorem 1]) In the above notation, $G/K$ is commutative if and only if each Poisson algebra $S(\mathfrak{n}_i)^{K^i}$ is commutative.*

Note that the Poisson algebra $S(\mathfrak{n}_i)^{K^i}$ is commutative for any $K$-invariant subspace $\mathfrak{w}_i \subset \mathfrak{n}/\mathfrak{n}'$, not necessary irreducible.

For convenience of the reader we present here the classification results of [22], [43] and [44]. All maximal commutative homogeneous spaces of Heisenberg type with $\mathfrak{n}/\mathfrak{n}'$ being an irreducible $K$-module are listed in Table 3.1. The following notation is used:

$\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{z} = \mathfrak{n}'$ is the centre of $\mathfrak{n}$;

$\mathbb{H}_0$ is the space of purely imaginary quaternions;

$\mathbb{C}^m \otimes \mathbb{H}^n$ is the tensor product over $\mathbb{C}$;

$\mathbb{H}^m \otimes \mathbb{H}^n$ is the tensor product over $\mathbb{H}$;

$H\Lambda^2\mathbb{D}^n$, where $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$, is the skew-Hermitian square of $\mathbb{D}$;

$HS_0^2\mathbb{H}^n$ is the space of Hermitian quaternion matrices of order $n$ with zero trace.

For all items of Table 3.1 the commutation operation $\mathfrak{w} \times \mathfrak{w} \mapsto \mathfrak{z}$ is uniquely determined by the condition of $K$-equivariance up to a conjugation by elements of the centraliser $Z_{\mathrm{SO}(\mathfrak{w})}(K)$. Notation $(\mathrm{U}_1\times)F$ means that $K$ can be either $F$ or $\mathrm{U}_1\times F$. The cases in which $\mathrm{U}_1$ is necessary are indicated in the column "$\mathrm{U}_1$". Some spaces are not maximal. This is indicated in the column "max".

Table 3.1.

| | $K$ | $\mathfrak{w}$ | $\mathfrak{z}$ | $\mathrm{U}_1$ | max |
|---|---|---|---|---|---|
| 1 | $\mathrm{SO}_n$ | $\mathbb{R}^n$ | $\Lambda^2\mathbb{R}^n = \mathfrak{so}_n$ | | |
| 2 | $\mathrm{Spin}_7$ | $\mathbb{R}^8$ | $\mathbb{R}^7$ | | |
| 3 | $G_2$ | $\mathbb{R}^7$ | $\mathbb{R}^7$ | | |
| 4 | $\mathrm{U}_1 \times \mathrm{SO}_n$ | $\mathbb{C}^n$ | $\mathbb{R}$ | | $n \neq 4$ |
| 5 | $(\mathrm{U}_1\times)\mathrm{SU}_n$ | $\mathbb{C}^n$ | $\Lambda^2\mathbb{C}^n \oplus \mathbb{R}$ | $n$ is odd | |
| 6 | $\mathrm{SU}_n$ ($n$ is odd) | $\mathbb{C}^n$ | $\Lambda^2\mathbb{C}^n$ | | |
| 7 | $\mathrm{SU}_n$ ($n$ is odd) | $\mathbb{C}^n$ | $\mathbb{R}$ | | |
| 8 | $\mathrm{U}_n$ | $\mathbb{C}^n$ | $H\Lambda^2\mathbb{C}^n = \mathfrak{u}_n$ | | |
| 9 | $(\mathrm{U}_1\times)\mathrm{Sp}_n$ | $\mathbb{H}^n$ | $HS^2_0\mathbb{H}^n \oplus \mathbb{H}_0$ | | |
| 10 | $\mathrm{U}_n$ | $S^2\mathbb{C}^n$ | $\mathbb{R}$ | | |
| 11 | $(\mathrm{U}_1\times)\mathrm{SU}_n$ ($n \geqslant 3$) | $\Lambda^2\mathbb{C}^n$ | $\mathbb{R}$ | $n$ is even | |
| 12 | $\mathrm{U}_1 \times \mathrm{Spin}_7$ | $\mathbb{C}^8$ | $\mathbb{R}^7 \oplus \mathbb{R}$ | | |
| 13 | $\mathrm{U}_1 \times \mathrm{Spin}_9$ | $\mathbb{C}^{16}$ | $\mathbb{R}$ | | |
| 14 | $(\mathrm{U}_1\times)\mathrm{Spin}_{10}$ | $\mathbb{C}^{16}$ | $\mathbb{R}$ | | |
| 15 | $\mathrm{U}_1 \times G_2$ | $\mathbb{C}^7$ | $\mathbb{R}$ | | |
| 16 | $\mathrm{U}_1 \times E_6$ | $\mathbb{C}^{27}$ | $\mathbb{R}$ | | |
| 17 | $\mathrm{Sp}_1 \times \mathrm{Sp}_n$ | $\mathbb{H}^n$ | $\mathbb{H}_0 = \mathfrak{sp}_1$ | | $n \geqslant 2$ |
| 18 | $\mathrm{Sp}_2 \times \mathrm{Sp}_n$ | $\mathbb{H}^2 \otimes \mathbb{H}^n$ | $H\Lambda^2\mathbb{H}^2 = \mathfrak{sp}_2$ | | |
| 19 | $(\mathrm{U}_1\times)\mathrm{SU}_m \times \mathrm{SU}_n$ ($m, n \geqslant 3$) | $\mathbb{C}^m \otimes \mathbb{C}^n$ | $\mathbb{R}$ | $m = n$ | |
| 20 | $(\mathrm{U}_1\times)\mathrm{SU}_2 \times \mathrm{SU}_n$ | $\mathbb{C}^2 \otimes \mathbb{C}^n$ | $H\Lambda^2\mathbb{C}^2 = \mathfrak{u}_2$ | $n = 2$ | |
| 21 | $(\mathrm{U}_1\times)\mathrm{SU}_n \times \mathrm{Sp}_2$ | $\mathbb{C}^n \otimes \mathbb{H}^2$ | $\mathbb{R}$ | $n \leqslant 4$ | $n \geqslant 3$ |
| 22 | $\mathrm{U}_2 \times \mathrm{Sp}_n$ | $\mathbb{C}^2 \otimes \mathbb{H}^n$ | $H\Lambda^2\mathbb{C}^2 = \mathfrak{u}_2$ | | |
| 23 | $\mathrm{U}_3 \times \mathrm{Sp}_n$ | $\mathbb{C}^3 \otimes \mathbb{H}^n$ | $\mathbb{R}$ | | $n \geqslant 2$ |

*Remark.* There is a small inaccuracy in tables of [43] and [44]. The homogeneous space $(N \rightthreetimes \mathrm{U}_n)/\mathrm{U}_n$, where $\mathfrak{n} = \mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n \oplus \mathbb{R}$, is commutative regardless of the parity of $n$.

Suppose $(N \rightthreetimes K)/K$ is a principal commutative homogeneous space and $\mathfrak{n}' \neq 0$. There is a non-Abelian subspace $\mathfrak{w}_1 \subset \mathfrak{n}/\mathfrak{n}'$. Denote by $K_e$ the maximal connected subgroup of $K$ acting on $\mathfrak{w}_1$ locally effectively. Then $K = K_e \times H$, where $H$ acts on $\mathfrak{n}_1$ trivially. The pair $(K_e, \mathfrak{n}_1)$ is either in item or a a central reduction of an item of Table 3.1.

The classification of maximal principal indecomposable $\mathrm{Sp}_1$-saturated commutative spaces of Heisenberg type is being done in the following way. For each commutative homogeneous space $(N_1 \rightthreetimes K_e)/K_e$ with $\mathfrak{n}_1/\mathfrak{n}_1' = \mathfrak{w}_1$ being an irreducible $K_e$-module, we find out all commutative space $(N \rightthreetimes K)/K$ such that $K = K_e \times H$, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{v}^1$. The classification tools are Theorem 3.1, Table 3.1, and the tables of all irreducible representation of simple Lie algebras with non-trivial generic stabilisers [14].

For example, if $K_e = \mathrm{SO}_n$, $\mathfrak{n}_1 = \mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n$, then $K = K_e$, $\mathfrak{n} = \mathfrak{n}_1$. Here the homogeneous space $(N_1 \rightthreetimes F)/F$, is not commutative for any proper subgroup $F \subset \mathrm{SO}_n$, see [3]. Hence, $\pi_e(K^1) = K_e$ and, by Lemma 1.7, either the action $\mathrm{SO}_n : \mathfrak{v}^1$ is trivial or $(N \rightthreetimes K)/K$ does not satisfy condition (3) of Definition 8.

We say that the action $K : \mathfrak{n}$ is *commutative* if the corresponding homogeneous space $(N \rightthreetimes K)/K$ is commutative.

**Lemma 3.2.** *Suppose that $K_e = K_e' \times \mathrm{U}_1$, $[\mathfrak{w}_1, \mathfrak{w}_1] \neq 0$ and $\mathfrak{w}_1 = W \otimes \mathbb{R}^2$, where $K_e'$ acts on $W$ and $\mathrm{U}_1$ on $\mathbb{R}^2$. Let $F$ be a proper subgroup of $K_e'$. If the action $F : W$ is reducible then $(N_1 \rightthreetimes (F \times \mathrm{U}_1))/(F \times \mathrm{U}_1)$ is not commutative.*

*Proof.* We show that the action of $H = (\mathrm{SO}_n \times \mathrm{SO}_m) \times \mathrm{SO}_2$ on $\mathfrak{n}_1 \cong (\mathbb{R}^n \oplus \mathbb{R}^m) \otimes \mathbb{R}^2 \oplus [\mathfrak{w}_1, \mathfrak{w}_1]$ cannot be commutative. Assume that it is commutative and apply Theorem 3.1. We have $H_*(\mathbb{R}^n \otimes \mathbb{R}^2)^0 = \mathrm{SO}_{n-1} \times \mathrm{SO}_m$. The subspace $\mathbb{R}^m \otimes \mathbb{R}^2$ is a sum of two isomorphic $\mathrm{SO}_{n-1} \times \mathrm{SO}_m$-modules. Hence, $\mathbb{R}^m \otimes \mathbb{R}^2$ is a commutative subalgebra of $\mathfrak{n}_1$. This can happen only if $[\mathfrak{w}_1, \mathfrak{w}_1] = 0$. $\square$

**Lemma 3.3.** *Let $(N \rightthreetimes K)/K$ be a commutative homogeneous space from row 1, 5, 6, 8, 9, 12, 18, 20 or 22 of Table 3.1. Suppose a subgroup $F \subset K$ acts on $\mathfrak{n}/\mathfrak{n}'$ reducibly. Then $(N \rightthreetimes F)/F$ is not commutative.*

*Proof.* Assume that $(N \rightthreetimes F)/F$ is commutative. Then due to [43, Prop. 15] there are at list two subspaces $V_1, V_2 \subset \mathfrak{n}/\mathfrak{n}'$, such that $V_1 \oplus V_2 = \mathfrak{n}/\mathfrak{n}'$ and $[V_1, V_2] = 0$. Evidently, this is not true in cases 1, 5, 6, 8. For the same reason, in cases 18, 20 and 22 $F$ contains the first simple factor of $K$, either $\mathrm{Sp}_2$ or $\mathrm{SU}_2$.

In case 9 $F$ has to be a subgroup of $Sp_m \times \mathrm{Sp}_{n-m}$. But subspaces $\mathbb{H}^m$ and $\mathbb{H}^{n-m}$ do not commute with each other.

Consider case 12. It follows form Lemma 3.2 and [43, Prop. 15] that $F = \mathrm{U}_1 \times H$, where $H \subset \mathrm{Spin}_7$ and the representation $H : \mathbb{R}^8$ is irreducible. Since $F : (\mathbb{R}^2 \otimes \mathbb{R}^8)$ is reducible, we have $H \subset \mathrm{SU}_4$. Now $(\mathfrak{n}/\mathfrak{n}') \cong \mathbb{C}^4 \oplus \mathbb{C}^4$ as an $F$-module. But these two subspaces do not commute with each other. This contradicts [43, Prop. 15].

In case 18, we have $F \subset \mathrm{Sp}_2 \times \mathrm{Sp}_m \times \mathrm{Sp}_{n-m}$, $F_*(\mathbb{H}^2 \otimes \mathbb{H}^m) \subset \mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_n$. The subspace $\mathbb{H}^2 \otimes \mathbb{H}^{n-m}$ is a sum of two isomorphic $F_*(\mathbb{H}^2 \otimes \mathbb{H}^m)$-modules. According to [43, Prop. 15], $\mathbb{H}^2 \otimes \mathbb{H}^{n-m}$ should be an Abelian subalgebra of $\mathfrak{n}$. But this is not so.

In case 20, $F$ is a subgroup of either $\mathrm{SU}_2 \times \mathrm{U}_m \times \mathrm{U}_{n-m}$ or $\mathrm{SU}_2 \times \mathrm{Sp}_{m/2}$ for even $m$. If $F \subset \mathrm{SU}_2 \times \mathrm{U}_m \times \mathrm{U}_{n-m}$, we apply the same reasoning as in case 18. If $F \subset \mathrm{SU}_2 \times \mathrm{Sp}_{m/2}$, then $\mathfrak{n}/\mathfrak{n}' = \mathbb{H}^{m/2} \oplus \mathbb{H}^{m/2}$ is a direct sum of two isomorphic $F$-modules. Hence by [43, Prop. 15], $\mathfrak{n}$ should be Abelian. The 22-d case is exactly the same. □

**Lemma 3.4.** *Let $F \subset \mathrm{Sp}_n$, $n \geq 2$ and $(F_*(\mathbb{H}^n))^0 = \{e\}$. Then the image of the generic stabiliser $(F \times \mathrm{Sp}_m)_*(\mathbb{H}^n \otimes \mathbb{H}^m)$ under the projection on $F$ is Abelian.*

*Proof.* Assume that this is not the case, i.e., the image contains $\mathrm{Sp}_1$. Stablisers of decomposable vectors are contained in a generic stabiliser $(F \times \mathrm{Sp}_m)_*(V)$ up to conjugation. Hence, the restriction $F|_{\xi\mathbb{H}}$ contains $\mathrm{Sp}_1$ for each non-zero $\xi \in \mathbb{H}^n$. Due to Lemma 1.7, we have $F = \mathrm{Sp}_n$. But then $F_*(\mathbb{H}^n) = \mathrm{Sp}_{n-1}$. □

Recall that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{v}^1$, $K = K_e \times H$, where $H$ acts on $\mathfrak{n}_1$ trivially. Denote by $\pi_e$ the natural projection $K \to K_e$.

**Lemma 3.5.** *Suppose $(N \rtimes K)/K$ is commutative, $[\mathfrak{n}_1, \mathfrak{n}_1] \neq 0$, and $\pi_e(K^1) = (\mathrm{U}_1)^n$. Then $\mathfrak{n}_1 = \mathbb{R}^{2n} \oplus \mathbb{R}$, $K_e = \mathrm{U}_n$.*

*Proof.* An irreducible representation of $\mathrm{U}_1$ on a real vector space is either trivial ($\mathbb{R}$) or $\mathbb{R}^2$. If $\mathfrak{w}_1$ is the direct sum of more than $n$ $K^1$-invariant summands, then two of them are isomorphic and there is a non-zero $\eta \in \mathfrak{w}_1$ such that $[\eta, \mathfrak{w}_1] = 0$. But $K_e\eta = \mathfrak{w}_1 \subset \mathfrak{z}(\mathfrak{n}_1)$. By the same reason $\mathfrak{w}_1^{(\mathrm{U}_1)^n} = 0$. Because the action $(U_1)^n : \mathfrak{w}_1$ is locally effective, $\mathfrak{w}_1 = \mathbb{R}^{2n}$. We have $\Lambda^2\mathbb{R}^2 = \mathbb{R}$, hence $K^1$ acts on $\mathfrak{n}_1'$ trivially. Each element of $K_e$ is contained in some maximal torus, that is up to conjugation in $\pi_e(K^1)$. Hence $K_e$ acts on $\mathfrak{n}_1'$ trivially and $K_e \subset \mathrm{U}_n$. The group $\mathrm{U}_n$ has no proper subgroups of rank $n$ acting on $\mathbb{R}^{2n}$ irreducibly. Thus we have $K_e = \mathrm{U}_n$, $\mathfrak{n}_1' = (\Lambda^2\mathbb{R}^n)^{\mathrm{U}_n} \cong \mathbb{R}$. □

From now on, let $(N \rtimes K)/K$ be an indecomposable maximal $\mathrm{Sp}_1$-saturated principal commutative space with $\mathfrak{n}_1 \neq \mathfrak{n}$. In particular, the connected centre $Z(K_e)$ of $K_e$ acts on $\mathfrak{v}^1$ trivially. Let $\mathfrak{a} \subset \mathfrak{n}$ be a $K$-invariant subalgebra. Clearly, if the action $K : \mathfrak{n}$ is commutative, then $K : \mathfrak{a}$ is also commutative. We assume that $K : \mathfrak{n}$ is not a "subaction" of some larger commutative action.

Decompose $\mathfrak{n}$ into a direct sum of $K$-invariant subspaces $\mathfrak{n} = \mathfrak{n}_1 \oplus V_2 \otimes_{\mathbb{D}_2} V^2 \oplus \ldots \oplus V_q \otimes_{\mathbb{D}_q} V^q \oplus V_{\mathrm{tr}}$, where $V_i$ are pairwise non-isomorphic irreducible non-trivial $K_e'$-modules, $V_{\mathrm{tr}}$ and $V^i$ are trivial $K_e$-modules, and $H$ acts on each $V_i$ trivially. In order to classify all commutative spaces $(N \rtimes K)/K$ with a given action $K_e : \mathfrak{n}_1$, we have to describe possible $V_i$, then dimensions of $V^i$, afterwards the actions $K : \bigoplus_{i=2}^{q} V_i \otimes V^i$ and $K : V_{\mathrm{tr}}$. Once again we use Élashvili's classification [14]. Note that, according to Lemma 3.5, $V_i$ could be isomorphic to $\mathfrak{k}_e'$ only if $(K_e, \mathfrak{n}_1) = (\mathrm{U}_n, \mathbb{C}^n \oplus \mathbb{R})$.

Assume for the time being that $K_e'$ is simple and denote it by $K_1$. (In general, $K_e'$ is a product of at most two simple direct factors.) Suppose $(K_1)_*(V_i)$ is finite for some $V_i$ such that $V_i \otimes_{\mathbb{D}_i} V^i \subset \mathfrak{v}^1$. Then $\pi_e(K^1)$ is Abelian. For $\mathbb{D}_i = \mathbb{R}$ or $\mathbb{C}$, the statement is clear. For $\mathbb{D}_i = \mathbb{H}$, it follows from Lemma 3.4. Thus, $(K_1)_*(V_i)$ is non-trivial (infinite) for all $V_i$, unless $(K_e, \mathfrak{n}_1) = (\mathrm{U}_2, \mathbb{C}^2 \oplus \mathbb{R})$.

First we consider pairs $(K_e, \mathfrak{n}_1)$ such that $\mathfrak{n}_1'$ is a non-trivial $K_e$-module.

*Example* 9. Let $(K_e, \mathfrak{n}_1)$ be the second item of Table 3.1. We show that $\mathfrak{n} \subset \mathfrak{n}_1 \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$. All representations of $\mathrm{Spin}_7$ are orthogonal, so here all $\mathbb{D}_i$ equal $\mathbb{R}$. The group $\mathrm{Spin}_7$ has only three irreducible representations with infinite generic stabiliser, namely $\mathfrak{so}_7$, $\mathbb{R}^7$ and $\mathbb{R}^8$. If $V_i = \mathbb{R}^8$ for some $i$, then $K^1$ has a non-zero invariant in $\mathfrak{w}_1$, which commute with $\mathfrak{w}_1$. This is a contradiction. Thus $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathbb{R}^7 \otimes V^2 \oplus V_{\mathrm{tr}}$. If $\dim V^2 \geq 3$, then $\pi_e(K^1) \subset \mathrm{SU}_2 \times \mathrm{SU}_2$. But the action $\mathrm{SU}_2 \times \mathrm{SU}_2 : (\mathbb{C}^2 \oplus \mathbb{C}^2) \oplus \mathbb{R}^7$ is not commutative. In case $\dim V^2 = 2$ we have $\pi_e(K^1) = \mathrm{Spin}_5 = \mathrm{Sp}_2$. The pair $(\mathrm{Sp}_2, \mathbb{H}^2 \oplus \mathbb{R}^7)$ is a central reduction of item 9 of Table 3.1 with $n = 2$ by a subgroup corresponding to $\mathbb{H}_0$ (here $HS_0^2\mathbb{H}^2 \cong \mathbb{R}^7$ as an $\mathrm{Sp}_2$-module).

Since $\dim V^2 \leq 2$, the maximal connected subgroup of $K$ acting on $\mathfrak{n}_1 \oplus \mathbb{R}^7 \otimes V^2$ locally effectively is either $\mathrm{Spin}_7$ or $\mathrm{Spin}_7 \times \mathrm{SO}_2$. Anyway, because $(N \rtimes K)/K$ is principal, this subgroup acts trivially on $V_{\mathrm{tr}}$. Hence, $V_{\mathrm{tr}} \subset \mathfrak{n}'$ and $K \subset \mathrm{Spin}_7 \times \mathrm{SO}_2$. Clearly, $V_{\mathrm{tr}} \cap \mathfrak{n}_1' = \{0\}$. Assume that there is a non-Abelian subspace $\mathfrak{w}_2 \subset \mathfrak{n}$. Then it is either $\mathbb{R}^7$ or $\mathbb{R}^7 \otimes \mathbb{R}^2$. The first case is not possible, because $\Lambda^2\mathbb{R}^7 \cong \mathfrak{so}_7$ as a $\mathrm{Spin}_7$-module. In the second case we apply Theorem 3.1 to $\mathfrak{w}_2 = \mathbb{R}^7 \otimes \mathbb{R}^2$. We have $(\mathrm{Spin}_7)_*(\mathbb{R}^8 \oplus \mathbb{R}^7) \subset \mathrm{Spin}_6$. Hence $K^2 \subset \mathrm{Spin}_6 \times \mathrm{SO}_2$. By Lemma 3.2, the action $\mathrm{SO}_6 \times \mathrm{SO}_2 : \mathfrak{w}_2 \oplus [\mathfrak{w}_2, \mathfrak{w}_2]$ is commutative only if $[\mathfrak{w}_2, \mathfrak{w}_2] = 0$. Thus, $\mathfrak{n}' = \mathfrak{n}_1'$, $V_{\mathrm{tr}} = 0$ and $\mathfrak{n} \subset \mathfrak{n}_1 \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$. The corresponding commutative space is indicated in the 13-th row of Table 3.2.

For convenience of the reader, we list all irreducible representations of $\mathfrak{su}_n$ with non-trivial generic stabiliser. They are described by the highest weights of the complexifications.

Table $A_{n-1}$.

| $n$ | representation | $(\mathrm{SU}_n)_*(V)^0$ |
|---|---|---|
| | $R(\varpi_1) \oplus R(\varpi_1)^*$ | $\mathrm{SU}_{n-1}$ |
| | $R(\varpi_2) \oplus R(\varpi_2)^*$ | $(\mathrm{SU}_2)^{[n/2]}$ |
| | $R(2\varpi_1) \oplus R(2\varpi_1)^*$ | $(\mathrm{U}_1)^{[n/2]}$ |
| | $R(\varpi_1 + \varpi_1^*)$ | $(\mathrm{U}_1)^{n-1}$ |
| 4 | $R(\varpi_2)$ | $\mathrm{Sp}_2$ |
| 6 | $2R(\varpi_3)$ | $(\mathrm{U}_1)^2$ |

Note that the action $(\mathrm{SU}_n)_*(V) : \mathbb{C}^n$ is irreducible only in one case: $n = 4$, $V = R(\varpi_2) \cong \mathbb{R}^6$.

There are 9 pirs $(K_e, \mathfrak{n}_1)$ such that $K_e$ has two simple direct factors. Namely, seven last items of Table 3.1 and their central reductions. Let $K_1 \lhd K_e$ be a simple normal subgroup. Similar to the case of simple $K_e'$, one can show that if $K_1 \neq \mathrm{SU}_2$, then $(K_1)_*(V_i)$ is infinite for each $V_i$ such that $V_i \otimes_{\mathbb{D}_i} V^i \subset \mathfrak{v}^1$.

According to Lemma 3.3, for items $1, 5, 6, 8, 9, 12, 18, 20, 22$ of Table 3.1, the action $K^1 : \mathfrak{w}_1$ have to be irreducible. This leaves only a few possibilities for $V_i$. The obtained commutative spaces are listed in rows 2, 4, 5 of Table 3.2.

The Lie group $G_2$ has only two irreducible representations with non-trivial generic stabiliser, namely adjoint one and $\mathbb{R}^7$. Thus, if $(K_e, \mathfrak{n}_1)$ is the pair from the 3-d row of Table 3.1, then $K = K_e$ and $\mathfrak{n} = \mathfrak{n}_1$.

Calculations in cases $((\mathrm{U}_1)\times\mathrm{Sp}_n, \mathbb{H}^n\oplus\mathbb{H}_0)$, $((\mathrm{U}_1)\times\mathrm{Sp}_n, \mathbb{H}^n\oplus\mathbb{R})$ and $(\mathrm{Sp}_1\times\mathrm{Sp}_n, \mathbb{H}^n\oplus\mathfrak{sp}_1)$ do not differ much. By our assumptions subgroups $\mathrm{U}_1$ and $\mathrm{Sp}_1$ act on $\mathfrak{v}^1$ trivially. The result is given in rows 8–12 of Table 3.2.

If $\mathfrak{n}'$ is a trivial $K$-module, the calculations are even simpler. However, we have more such cases. Recall that $\mathfrak{w}_1(\mathbb{C}) = W_1 \oplus W_1^*$ as a $K$-module. We have to check whether the action $\pi_e(K^1) : W_1$ is spherical or not. If $(K_e, \mathfrak{n}_1)$ is item $10, 13, 15, 16, 21, 23$, 4 with $n \neq 8$ or a central reduction of item 22 of Table 3.1, then $K = K_e$, $\mathfrak{n} = \mathfrak{n}_1$. One can prove it using tables of [14] and in some cases Lemma 3.2. For all other pairs $(K_e, \mathfrak{n}_1)$ with $\mathfrak{n}_1'$ being trivial $K$-module, $\mathfrak{n}$ can be larger than $\mathfrak{n}_1$. We will consider one typical example in full details. Other cases are very similar to it.

*Example* 10. Here we prove that all principal $\mathrm{Sp}_1$-saturated maximal commutative pairs $(K, \mathfrak{n})$ with $(K_e, \mathfrak{n}_1) = ((\mathrm{U}_1\times)\mathrm{SU}_n, \mathbb{C}^n \oplus \mathbb{R})$ and $K \neq K_e$ are items $1, 3, 7, 19, 24$, and 25 in Table 3.2. Commutativity of all items of Table 3.2 is proved below, see Theorem 3.6. Recall our notation: $K = K_e \times H$, $\mathfrak{n}/\mathfrak{n}' = \bigoplus_{i=1}^{p} \mathfrak{w}_i$, $\mathfrak{n}_1 = \mathfrak{w}_1 \oplus [\mathfrak{w}_1, \mathfrak{w}_1]$, and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{v}^1$.

First suppose that $n = 2$. Let $\mathfrak{w}_2 \subset \mathfrak{v}^1 \cap (\mathfrak{n}/\mathfrak{n}')$ be an irreducible $K$-invariant subspace on which $\mathrm{SU}_2$ acts non-trivially. Then $\mathfrak{w}_2 = V_2 \otimes_{\mathbb{D}} V^2$, where $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $V_2$ is an $\mathrm{SU}_2$-module, and $V^2$ is an $H$-module. Assume that $\mathbb{D} = \mathbb{H}$. Then $\dim V_2 > 1$ due to condition (3) of Definition 8 and $\pi_e(K_*(\mathfrak{w}_2)) = \{e\}$.

Thus $\mathbb{D}$ is either $\mathbb{R}$ or $\mathbb{C}$, $\pi_e(K_*(\mathfrak{w}_2)) = \mathrm{U}_1$ up to a local isomorphism, and $\mathrm{SU}_2$ acts trivially on $\mathfrak{w}_3 \oplus \ldots \oplus \mathfrak{w}_p$. In case $\mathbb{D} = \mathbb{R}$, we get $V_2 = \mathbb{R}^3$, $V^2 = \mathbb{R}$, and, hence, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{su}_2$.

In case $\mathbb{D} = \mathbb{C}$, we have $V_2 = \mathbb{C}^2$. Let $\mathfrak{v}^1 = \mathfrak{w}_2 \oplus V$ be a $K$-invariant decomposition. Then $\pi_e((\mathrm{SU}_2\times H_*(V))_*(\mathfrak{w}_2)) = \mathrm{U}_1$. Hence, $H_*(V)$ acts on a generic subspace $\mathbb{C}^2 \subset V^2$ as $(\mathrm{S})\mathrm{U}_2$. In particular, $H_*(V)$ is transitive on a $(2m-1)$-dimensional sphere, where $m = \dim V^2$. It follows, see e.g. [32], that $H_*(V)$ acts on $V^2$ as one of the following groups: $\mathrm{U}_m, \mathrm{SU}_m, \mathrm{U}_1\times\mathrm{Sp}_{m/2}, \mathrm{Sp}_{m/2}$. Thus, either $V$ is a trivial $K$-module and $H$ is one of the groups: $\mathrm{U}_m, \mathrm{SU}_m, \mathrm{U}_1\times\mathrm{Sp}_{m/2}, \mathrm{Sp}_{m/2}$; or $H = (\mathrm{S})\mathrm{U}_4$, $V = \mathbb{R}^6$. These commutative homogeneous spaces are items $7, 19, 24$, and 25 of Table 3.2.

Suppose now that $n > 2$. Here $(\mathfrak{su}_n)_*(V_i)$ is non-trivial for each $V_i$. Moreover, according to Lemma 3.5 and Table $A_{n-1}$, if $(\mathfrak{su}_n)_*(V_i)$ is Abelian, then $V_i = \mathfrak{su}_n$ and $V^i = \mathbb{R}$. Thus, each $V_i$ is one of the following three spaces: $\mathfrak{su}_n$; $\mathbb{C}^n$; $\Lambda^2\mathbb{C}^n$ with $n > 4$ and $\mathbb{R}^6$ in case $n = 4$. Suppose $\mathfrak{su}_n \subset \mathfrak{v}^1$ is an irreducible $K$-invariant subspace. Let $(\mathrm{U}_1)^{n-1} \subset \mathrm{SU}_{n-1}$ be a maximal torus. Then the action $(\mathrm{U}_1)^n : (\mathfrak{v}^1/\mathfrak{su}_n)$ is trivial due to Lemma 3.5. Thus $\mathfrak{v}^1 = \mathfrak{su}_n$. This commutative space is the first item of Table 3.2. Below, we assume that $\mathfrak{su}_n$

is not contained in $\mathfrak{v}^1$.

Consider case $n = 4$. We have $\mathfrak{v}^1 = \mathbb{C}^4 \otimes V^2 \oplus \mathbb{R}^6 \otimes \mathbb{R}^s \oplus V_{\mathrm{tr}}$. Note that $(\mathrm{SU}_4)_*(\mathbb{R}^6 \otimes \mathbb{R}^3)$ is finite, so $s \leq 2$. Also $(\mathrm{U}_1 \times \mathrm{U}_4)_*(\mathbb{C}^4 \oplus \mathbb{R}^6 \otimes \mathbb{R}^2) = (\mathrm{U}_1)^3$ and $\mathbb{C}^4$ is not a spherical representation of $(\mathbb{C}^*)^3$. Hence, if $s = 2$, then $(K, \mathfrak{n}) = ((\mathrm{S})\mathrm{U}_4(\times \mathrm{SO}_2), \mathbb{C}^4 \oplus \mathbb{R} \oplus \mathbb{R}^6 \otimes \mathbb{R}^2)$. If $s = 1$, then $\dim V^2 = 1$ and $(K, \mathfrak{n}) = (\mathrm{U}_4 \times \mathrm{U}_1, (\mathbb{C}^4 \oplus \mathbb{R}) \oplus (\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6)$. If $s = 0$, then $\mathfrak{v}^1 = \mathbb{C}^n \otimes V^2 \oplus V_{\mathrm{tr}}$. This last possibility is the same for general $n$ and is dealt upon below.

Note that $(\mathrm{SU}_n)_*(\Lambda^2\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n) = \mathrm{U}_1$ and $(\mathrm{SU}_n)_*(\Lambda^2\mathbb{C}^n \oplus \mathbb{C}^n) = \{e\}$. Thus, either $\mathfrak{n} = \mathfrak{n}_1 \oplus (\Lambda^2\mathbb{C}^n \oplus \mathbb{R})$ or $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathbb{C}^n \otimes_{\mathbb{D}_2} V^2 \oplus V_{\mathrm{tr}}$. Here $\mathbb{D}_2$ equals $\mathbb{C}$ or $\mathbb{R}$. If $\mathbb{D}_2 = \mathbb{R}$ and $\dim V^2 > 1$, then $\pi_e(K^1)$ is contained in $\mathrm{U}_1 \times \mathrm{U}_{n-2} \subset \mathrm{U}_2 \times \mathrm{U}_{n-2}$. Evidently, the $(\mathrm{U}_1 \times \mathrm{U}_{n-2})$-module $\mathbb{C}^n$ is not spherical. Hence, $\mathbb{D}_2 = \mathbb{C}$.

Suppose that $H_*(V_{\mathrm{tr}})$ acts on $V^2$ as $F \subset \mathrm{U}_r$, where $r = \dim V^2$. Set $d := \min(n, r)$. Then $\pi_e(K^1) = \pi_e((\mathrm{U}_n \times F)_*(\mathbb{C}^n \otimes \mathbb{C}^r)) \subset (\mathrm{U}_1)^d \times \mathrm{U}_{n-d}$. Since the action $K^1 : \mathfrak{n}_1$ is commutative, $\pi_e(K^1)$ contains $(\mathrm{U}_1)^d \times \mathrm{SU}_{n-d}$. It follows that $F$ acts on a generic subspace $\mathbb{C}^2 \subset V^2$ as $(\mathrm{S})\mathrm{U}_2$. Also, if $r > 2$, then $F$ acts on a generic subspace $\mathbb{C}^3 \subset V^2$ as $(\mathrm{S})\mathrm{U}_3$. Thus $F = (\mathrm{S})\mathrm{U}_r$, $H_*(V_{\mathrm{tr}}) = H$, and, hence, $V_{tr}$ is a trivial $K$-module. We conclude that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathbb{C}^2 \otimes \mathbb{C}^s \oplus \mathbb{R}$. This commutative spaces is item 3 of Table 3.2.

**Theorem 3.6.** *All indecomposable* $\mathrm{Sp}_1$*-saturated maximal principal commutative homogeneous spaces* $(N \rtimes K)/K$ *such that* $\mathfrak{n} \neq 0$ *and* $\mathfrak{n}/\mathfrak{n}'$ *is a reducible* $K$*-module are presented in Table 3.2 (in the sense that* $\mathfrak{n}$ *is a* $K$*-invariant subalgebra of* $\mathfrak{n}_{max}$*).*

Explanations to Table 3.2. The algebra $\mathfrak{n}_{\max}$ is described in the following way. Each subspace in parentheses represents a subalgebra $\mathfrak{w}_i \oplus [\mathfrak{w}_i, \mathfrak{w}_i]$. The spaces given outside parentheses are Abelian. The actions $K : \mathfrak{n}_{\max}$ are uniquely determined by the condition that representations $K : \mathfrak{w}_i$ are irreducible. Notation $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2})$ means that this normal subgroup of $K$ can be equal to either of these three groups. Appearance of the symbol $\mathrm{Sp}_{n/2}$ means that $n$ is even.

*Proof.* It was already explained that all such commutative spaces are contained in Table 3.2. Now using Theorem 3.1, Table 3.1, and the list of the spherical representations from [24], we prove that all these spaces are commutative.

It is proved in [3] that the spaces contained in rows $3, 7, 8, 16, 19$ and $20$ are commutative.

Suppose $\mathfrak{n}$ contains an Abelian $K$-invariant ideal $\mathfrak{a}$. According to Theorem 3.1, $K : \mathfrak{n}$ is commutative if and only if $K_*(\mathfrak{a}) : \mathfrak{n}/\mathfrak{a}$ is commutative. For items $2, 4, 5$ of Table 3.2 take $\mathfrak{a} = \mathbb{R}^6$. Then $K_*(\mathfrak{a}) : \mathfrak{n}/\mathfrak{a}$ appears in Table 3; hence, these three spaces are commutative. For items $6, 21, 24$, the pairs $K_*(\mathbb{R}^6) : \mathfrak{n}/\mathfrak{a}$, where $\mathfrak{a} = \mathbb{R}^6$ correspond to spherical representations. Analogously, for item 23 pair $K_*(\mathfrak{a}) : \mathfrak{n}/\mathfrak{a}$, where $\mathfrak{a} = \mathbb{R}^6 \oplus \mathbb{R}^6$, corresponds to a spherical representation.

Let $(N \rtimes L)/K$ be commutative. Consider the $K$-module $\mathfrak{l}/\mathfrak{k}$ as Abelian Lie algebra. Then $\mathfrak{n} \oplus (\mathfrak{l}/\mathfrak{k})$ is a 2-step nilpotent Lie algebra, and it follows from Theorems 1.3 and 3.1 that the action $K : \mathfrak{n} \oplus (\mathfrak{l}/\mathfrak{k})$ is commutative. The pairs in rows $1, 12$, and $15$ are obtained in this way from the commutative spaces $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$, $(H_{2n} \rtimes \mathrm{U}_{2n})/\mathrm{Sp}_n$, and $(H_8 \rtimes (\mathrm{SO}_8 \times \mathrm{SO}_2))/(\mathrm{Spin}_7 \times \mathrm{SO}_2)$, respectively. Since one obtains commutative homogeneous space of Euclidian type in case $[\mathfrak{n}, \mathfrak{n}] = 0$, these are the only non-trivial examples given by this construction.

Table 3.2.

| | $K$ | $\mathfrak{n}_{\max}$ |
|---|---|---|
| 1 | $\mathrm{U}_n$ | $(\mathbb{C}^n \oplus \mathbb{R}) \oplus \mathfrak{su}_n$ |
| 2 | $\mathrm{U}_4$ | $(\mathbb{C}^4 \oplus \Lambda^2\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 3 | $\mathrm{U}_1 \times \mathrm{U}_n$ | $(\mathbb{C}^n \oplus \mathbb{R}) \oplus (\Lambda^2\mathbb{C}^n \oplus \mathbb{R})$ |
| 4 | $\mathrm{SU}_4$ | $(\mathbb{C}^4 \oplus HS^2\mathbb{H}^2 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 5 | $\mathrm{U}_2 \times \mathrm{U}_4$ | $(\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus H\Lambda^2\mathbb{C}^2) \oplus \mathbb{R}^6$ |
| 6 | $\mathrm{SU}_4 \times \mathrm{U}_m$ | $(\mathbb{C}^4 \otimes \mathbb{C}^m \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 7 | $\mathrm{U}_m \times \mathrm{U}_n$ | $(\mathbb{C}^m \otimes \mathbb{C}^n \oplus \mathbb{R}) \oplus (\mathbb{C}^m \oplus \mathbb{R})$ |
| 8 | $\mathrm{U}_1 \times \mathrm{Sp}_n \times \mathrm{U}_1$ | $(\mathbb{H}^n \oplus \mathbb{R}) \oplus (\mathbb{H}^n \oplus \mathbb{R})$ |
| 9 | $\mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{U}_1$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus (\mathbb{H}^n \oplus \mathbb{R})$ |
| 10 | $\mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{Sp}_1$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus (\mathbb{H}^n \oplus \mathbb{H}_0)$ |
| 11 | $\mathrm{Sp}_n \times (\mathrm{Sp}_1, \mathrm{U}_1, \{e\}) \times \mathrm{Sp}_m$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus \mathbb{H}^n \otimes \mathbb{H}^m$ |
| 12 | $\mathrm{Sp}_n \times (\mathrm{Sp}_1, \mathrm{U}_1, \{e\})$ | $(\mathbb{H}^n \oplus \mathbb{H}_0) \oplus HS_0^2\mathbb{H}^n$ |
| 13 | $\mathrm{Spin}_7 \times (\mathrm{SO}_2, \{e\})$ | $(\mathbb{R}^8 \oplus \mathbb{R}^7) \oplus \mathbb{R}^7 \otimes \mathbb{R}^2$ |
| 14 | $\mathrm{U}_1 \times \mathrm{Spin}_7$ | $(\mathbb{C}^7 \oplus \mathbb{R}) \oplus \mathbb{R}^8$ |
| 15 | $\mathrm{U}_1 \times \mathrm{Spin}_7$ | $(\mathbb{C}^8 \oplus \mathbb{R}) \oplus \mathbb{R}^7$ |
| 16 | $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{Spin}_8$ | $(\mathbb{C}^8_+ \oplus \mathbb{R}) \oplus (\mathbb{C}^8_- \oplus \mathbb{R})$ |
| 17 | $\mathrm{U}_1 \times \mathrm{Spin}_{10}$ | $(\mathbb{C}^{16} \oplus \mathbb{R}) \oplus \mathbb{R}^{10}$ |
| 18 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2}) \times \mathrm{SU}_2$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathfrak{su}_2$ |
| 19 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2}) \times \mathrm{U}_2$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \oplus \mathbb{R})$ |
| 20 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2}) \times \mathrm{SU}_2 \times$ $\times (\mathrm{SU}_m, \mathrm{U}_m, \mathrm{U}_1 \times \mathrm{Sp}_{m/2})$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^m \oplus \mathbb{R})$ |
| 21 | $(\mathrm{SU}_n, \mathrm{U}_n, \mathrm{U}_1 \times \mathrm{Sp}_{n/2}) \times \mathrm{SU}_2 \times \mathrm{U}_4$ | $(\mathbb{C}^n \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 22 | $\mathrm{U}_4 \times \mathrm{U}_2$ | $\mathbb{R}^6 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathfrak{su}_2$ |
| 23 | $\mathrm{U}_4 \times \mathrm{U}_2 \times \mathrm{U}_4$ | $\mathbb{R}^6 \oplus (\mathbb{C}^4 \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 24 | $\mathrm{U}_1 \times \mathrm{U}_1 \times \mathrm{SU}_4$ | $(\mathbb{C}^4 \oplus \mathbb{R}) \oplus (\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6$ |
| 25 | $(\mathrm{U}_1 \times)\mathrm{SU}_4(\times \mathrm{SO}_2)$ | $(\mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6 \otimes \mathbb{R}^2$ |

In the remaining nine cases we use Theorem 3.1. For instance, take item 11 with $K = \mathrm{Sp}_n \times \mathrm{Sp}_m$. Here $\mathfrak{n}$ contains only one non-Abelian subspace $\mathfrak{w}_1 \cong \mathbb{H}^n$. Set $d = |n - m|$ and $s = \min(n, m)$. Then $K^1 = K_*(\mathbb{H}^n \times \mathbb{H}^m) = (\mathrm{Sp}_1)^d \times \mathrm{Sp}_s$. Anyway, $(\mathrm{Sp}_1)^n$ is a subgroup of $\pi_e(K^1)$. To conclude, observe that the action $Sp_1 : (\mathbb{H} \oplus \mathbb{H}_0)$ is commutative according to Table 3.1. □

# Chapter 4

# Final classification

## 4.1 Trees and forests

We use notation of previous sections. As we have seen in Section 1.5, in order to classify all commutative spaces, one has to describe triples $(F, \check{F}, V)$ such that $\mathfrak{f} \subset \mathfrak{so}(V)$,

$$F = \mathrm{Sp}_1 \times \check{F} \text{ and } F = F_*(V)\check{F}. \qquad (**)$$

Suppose $F = (F_1 \times F_2)$ is acting on $V$. Then we denote by $(F_1)_\circledast(V)$ the image of $F_*(V)$ under the natural projection $F \to F_1$. Recall that $(F_1)_*(V)$ is a normal subgroup of $(F_1)_\circledast(V)$ (see Lemma 1.8). For $F = \mathrm{Sp}_1 \times \check{F}$, condition $F = F_*(V)\check{F}$ is equavalent to $(\mathrm{Sp}_1)_\circledast(V) = \mathrm{Sp}_1$. We assume that the triple $(F, \check{F}, V)$ is indecomposable, i.e., there is no decomposition $\check{F} = F_1 \cdot F_2$, $V = V_1 \oplus V_2$ such that $\mathrm{Sp}_1 \times F_1$ acts on $V_2$ trivially and $F_2$ acts trivially on $V_1$.

Consider a rooted tree $T_q$ with vertices $0, 1, \ldots, q$, where 0 is the root. To each vertex $i$ we attach a weight $d(i)$, which is either a positive integer $d(i)$ or $\infty$. Assume that $d(0) = 1$, each vertex $i$ with $d(i) = \infty$ has degree 1 and if $(i, j)$ is an edge with $d(j) = \infty$, then $d(i) > 1$. We say that a vertex $i$ is *finite* if $d(i) < \infty$ and an edge $(i, j)$ is finite if both $i$ and $j$ are finite. Let $F$ be a product of $\mathrm{Sp}_{d(i)}$ over all finite vertices, and $\check{F}$ be a product of $\mathrm{Sp}_{d(i)}$ over all finite vertices except the root. To each finite edge $(i, j)$ we attach a vector space $W_{i,j} := \mathbb{H}^{d(i)} \otimes_{\mathbb{H}} \mathbb{H}^{d(j)}$ and to an edge $(i, j)$ with $d(j) = \infty$, a vector space $W_{i,j} := HS_0^2\mathbb{H}^{d(i)}$. Note that, since $T_q$ is connected, if $d(j) = \infty$, then $d(i) < \infty$ for the single vertex $i$ connected with $j$ by an edge. Let $V$ be a direct sum of $W_{i,j}$ over all edges.

The group $\mathrm{Sp}_{d(i)}$ naturally acts on a subspace $(\bigoplus\limits_{(i,j)} W_{i,j}) \subset V$. This gives rise to a representation $F : V$ and to an embedding $\mathfrak{f} \subset \mathfrak{so}(V)$. For example, a tree with two vertices corresponds to a linear representation $\mathrm{Sp}_1 \times \mathrm{Sp}_{d(1)} : \mathbb{H}^{d(1)}$.

**Lemma 4.1.** *The indecomposable triples $(F, \check{F}, V)$, where $F = \mathrm{Sp}_1 \times \check{F}$, $(\mathrm{Sp}_1)_\circledast(V) = \mathrm{Sp}_1$ are in one-to-one correspondence with the described above rooted trees $T_q$ such that*

*(I) if $d(i) > 1$, then the vertex $i$ has degree at most 2;*

*(II) if $d(i) > 1$, $d(j) > 1$ and vertices $i$ and $j$ are connected by an edge, then one of then has degree 1.*

We have to make a few preparations before we give a proof. First of all, denote vertices by a corresponding numbers $d(i)$. Thus, the triple from Example 8 corresponds to the following tree.

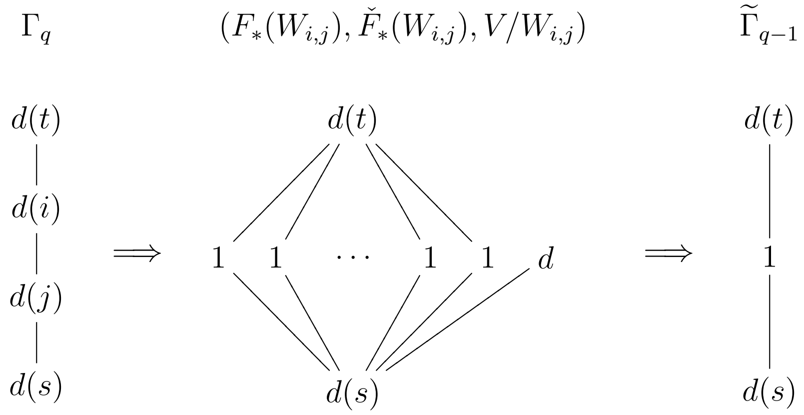
$$1 \text{ --- } 1 \text{ --- } 1 \text{ --- } 1 \text{ --- } \cdots \text{ --- } 1 \text{ --- } 1 \text{ --- } 1$$

Here the root is the first vertex, but it can be any of them. A generic stabiliser of $(\mathrm{Sp}_1)^n : n\mathbb{H}$ equals to $\mathrm{Sp}_1$ embedded diagonally in $(\mathrm{Sp}_1)^n$.

Suppose we have a weighted graph $\Gamma_q$, i.e., to each vertex $i$ of $\Gamma$ we have attached either a positive integer $d(i)$ or have set $d(i) := \infty$. Suppose there is at list one vertex with $d(i) = 1$ and each infinite vertex has degree 1. Then we can choose one vertex $i$ of $\Gamma_q$ with $d(i) = 1$ as a root and constructed a triple $(F, \check{F}, V)$ by the same principle as for a tree. Let $(i, j)$ be an edge of $\Gamma_q$ such that neither $i$ nor $j$ is the root. Denote by $\widetilde{\Gamma}_{q-1}$ the graph obtained from $\Gamma_q$ by contracting $(i, j)$ to a vertex of weight 1. Informaly speaking, we erase the edge $(i, j)$ and replace vertices $i, j$ by one vertex with weight 1. The new vertex is connected by edges with all old vertices which were connected by edges with either $i$ or $j$.

Let $W_{i,j} \subset V$ be an $F$-invariant subspace corresponding to an edge $(i, j)$. If $(F, \check{F}, V)$ satisfies condition $(**)$, then $(\mathrm{Sp}_1)_{\circledast}(V_1) = \mathrm{Sp}_1$ for any $F$-invariant subspace $V_1 \subset V$, i.e., the triple $(F, \check{F}, V_1)$ also satisfies condition $(**)$. In particular, $(\mathrm{Sp}_1)_{\circledast}(W_{i,j}) = \mathrm{Sp}_1$ and $F_*(V/W_{i,j}) = \mathrm{Sp}_1 \times \check{F}_*(V/W_{i,j})$. Thus, if the triple $(F, \check{F}, V)$ satisfies condition $(**)$, then $(F_*(W_{i,j}), \check{F}_*(W_{i,j}), V/W_{i,j})$ also does.

Recall that a direct factor $\mathrm{Sp}_{d(r)}$ acts on $W_{i,j}$ trivially if $r \neq i, j$. Suppose that $d(i) \leq d(j)$. Set $d = d(j) - d(i)$, if $d(j) = \infty$, we assume that $d = 0$. Then $(\mathrm{Sp}_{d_i} \times \mathrm{Sp}_{d_j})_*(W_{i,j}) = (\mathrm{Sp}_1)^{d(i)} \times \mathrm{Sp}_d$. We illustrate the passage from $\Gamma_q$ and corresponding triple $(F, \check{F}, V)$ to $(F_*(W_{i,j}), \check{F}_*(W_{i,j}), V/W_{i,j})$ and then to $\widetilde{\Gamma}_{q-1}$ by the following picture.




Picture 1.

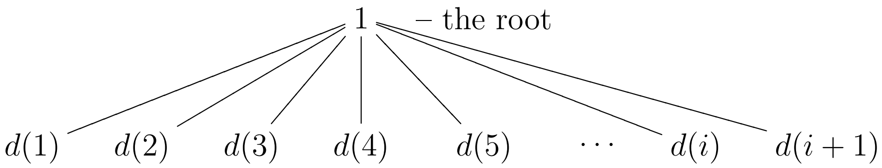
Here $d(t)$ and $d(s)$ are vertices connected with $d(i)$ and $d(j)$. In the second diagram we have either $d(i)+1$ or $1$ (if $d(i)=d(j)=1$) new vertices instead of two old ones; and in the third one we choose one vertex among $d(i)$ new ones. Note that, in case $d(j)=\infty$, we have no vertex $s$.

**Lemma 4.2.** *Suppose a triple $(F, \check{F}, V)$ corresponds to a graph $\Gamma_q$ and satisfies condition $(**)$. Then the triple corresponding to $\widetilde{\Gamma}_{q-1}$ also satisfies it.*

*Proof.* Assume that $d(i) \le d(j)$. When we replace $(F, \check{F}, V)$ by $(F_*(W_{i,j}), \check{F}_*(W_{i,j}), V/W_{i,j})$, we remove the vertices $i$, $j$ and the edge $(i,j)$ from $\Gamma_q$ and add $d(i)+1$ or $d(i)$ new vertices and several new edges, as shown on Picture 1. Let $\Gamma'_q$ be the graph corresponding to $F_*(W_{i,j}) : V/W_{i,j}$. Then $\widetilde{\Gamma}_{q-1}$ is a subgraph of $\Gamma'_q$. It contains the root of $\Gamma_q$ (which is also the root of $\Gamma'_q$) and corresponds to an $F_*(W_{i,j})$-invariant subspace in $V/W_{i,j}$. Thus, the triple corresponding to $\widetilde{\Gamma}_{q-1}$ satisfies condition $(**)$. $\square$

**Proof of Lemma 4.1.** Let $(F, \check{F}, V)$ be an indecomposable triple satisfying condition $(**)$. We construct a graph $\Gamma_q$ corresponding to it. We start with the root (the vertex 0) which has a weight $d(0)=1$. This vertex corresponds to $\mathrm{Sp}_1$ direct factor of $F$.

Let $W \subset V$ be an irreducible $F$-invariant subspace. Suppose $\mathrm{Sp}_1$ acts on $W$ non-trivially. Then according to Lemma 1.7, $W \cong \mathbb{H}^1 \otimes \mathbb{H}^n$ and $F$ acts on $\mathbb{H}^n$ as $\mathrm{Sp}_n$, where $\mathrm{Sp}_n \lhd F$. We put a vertex with a weight $n$ in $\Gamma_q$ and an edge connecting it with the root. By this procedure we construct the first level (all vertices connected with the root) of $\Gamma_q$. Let us check, that it has no double edges. We have $(\mathrm{Sp}_1 \times \mathrm{Sp}_n)_*(\mathbb{H}^n \oplus \mathbb{H}^n) = \mathrm{U}_1 \times \mathrm{Sp}_{n-2}$. Thus, if the graph contained a double edge, then $(\mathrm{Sp}_1)_{\circledast}(V) \subset \mathrm{U}_1$. Here is a picture of "the first level" part of the graph.




Denote by $W_1 \subset V$ the maximal $F$-invariant subspace such that $W_1^{\mathrm{Sp}_1} = 0$, i.e., $W_1 = \bigoplus W_{0,j}$, where the sum is taken over all edges $(0,j)$. Clearly, $W_1$ is an $F$-invariant complement of $W^1 := V^{\mathrm{Sp}_1}$.

Suppose a vertex $i$ with $d(i)=n$ is connected by an edge with the root. Then there is an $F$-invariant subspace $\mathbb{H}^n \subset V$, on which both $\mathrm{Sp}_n$ and $\mathrm{Sp}_1$ act non-trivially. We have $(\mathrm{Sp}_1 \times \mathrm{Sp}_n)_*(\mathbb{H}^n) = H_1 \times H_2$, where $H_1 \cong \mathrm{Sp}_1 \subset \mathrm{Sp}_1 \times \mathrm{Sp}_n$ and $H_2 \cong \mathrm{Sp}_{n-1} \subset \mathrm{Sp}_n$. Assume that the action $\mathrm{Sp}_n : W^1$ is non-trivial and take some irreducible $F$-invariant subspace $W \subset W^1$ which is a non-trivial $\mathrm{Sp}_n$-module. Let $\mathrm{Sp}_n \times H$ be the maximal connected subgroup of $F$ acting on $W$ locally effectively. Clearly, the action $(H_1 \times H_2) \times H$ on $W$ satisfies conditions of Lemma 1.7. Thus, if $H$ is non-trivial, then $H = \mathrm{Sp}_m$, $W = \mathbb{H}^r \otimes \mathbb{H}^m$ and the restriction $\mathbb{H}^r|_{H_1}$

contains only one-dimensional (over $\mathbb{H}$) representations of $\mathrm{Sp}_1$. Moreover, if $(\mathbb{H}^r)^{\mathrm{Sp}_1} \neq \mathbb{H}^{r-1}$, then $(H_1)_{\circledast}(W) \subset \mathrm{U}_1$ and also $(\mathrm{Sp}_1)_{\circledast}(V) \subset \mathrm{U}_1$. It follows that $r = n$.

The restriction of $\mathrm{Sp}_n : \mathbb{H}^n$ to $(\mathrm{Sp}_n)_{\circledast}(W^1)$ have to be a sum of $(\mathrm{Sp}_{n_i} : \mathbb{H}^{n_i})$ where $\sum n_i = n$. If $H$ is trivial, then $W$ is an irreducible representation of $\mathrm{Sp}_n$ and $(\mathrm{Sp}_n)_{\circledast}(W^1) \subset (\mathrm{Sp}_n)_*(W)$. In particular, $(\mathrm{Sp}_n)_*(W)$ contains $(\mathrm{Sp}_1)^n$. According to [14], $W = HS_0^2\mathbb{H}^n$ and $(\mathrm{Sp}_n)_*(W) = (\mathrm{Sp}_1)^n$. In this case, we inset an infinite vertex $j$, which corresponds not to a direct factor of $F$, but to a subspace $HS_0^2\mathbb{H}^n \subset V$. Clearly, $j$ has degree 1.

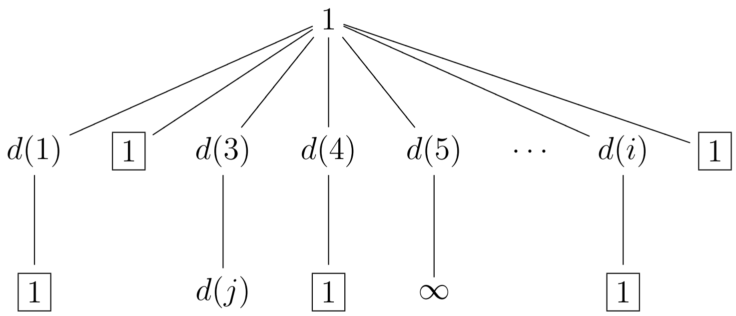
Assume that $n > 1$ and $W^1$ contains two different non-trivial $\mathrm{Sp}_n$-modules $\mathbb{H}^n \otimes \mathbb{H}^m$ and $\mathbb{H}^n \otimes \mathbb{H}^l$. We make another calculation:

$$(\mathrm{Sp}_n \times \mathrm{Sp}_m \times \mathrm{Sp}_l)_*(\mathbb{H}^n \otimes \mathbb{H}^m \oplus \mathbb{H}^n \otimes \mathbb{H}^l) \subset (\mathrm{Sp}_1 \times \mathrm{Sp}_{n-2}) \times \mathrm{Sp}_m \times \mathrm{Sp}_l,$$

where $\mathrm{Sp}_1 \subset \mathrm{Sp}_1 \times \mathrm{Sp}_1 \subset \mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_{n-2} \subset \mathrm{Sp}_n$. Clearly, $(\mathrm{Sp}_n)_{\circledast}(W^1) \subset \mathrm{Sp}_1 \times \mathrm{Sp}_{n-2}$, which is not allowed. Similar, $W^1$ cannot contain two copies of $HS_0^2\mathbb{H}^n$ or $HS_0^2\mathbb{H}^n \oplus \mathbb{H}^n \otimes \mathbb{H}^m$.

By the same reasoning, if $\mathrm{Sp}_m$ acts non-trivially on some other irreducible $F$-invariant subspaces, then it is of the form $\mathbb{H}^m \otimes \mathbb{H}^l$. Another calculation shows that if both $n, m > 1$ then $\mathrm{Sp}_m$ acts trivially on $V/(\mathbb{H}^n \otimes \mathbb{H}^m)$.

So far we have constructed a graph of the following type.

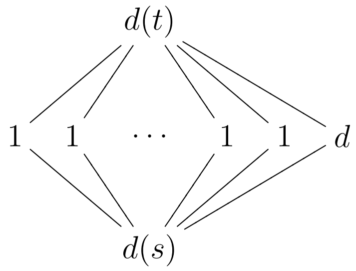



On this picture all integers $d(s)$ are greater then 1. Boxes around vertices with weights 1 are drawn because we have not described the actions of the corresponding groups $\mathrm{Sp}_1$.

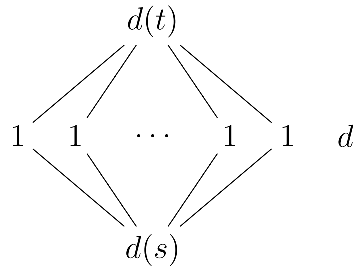
Let $H \cong \mathrm{Sp}_1$ be a normal subgroup corresponding to a vertex $a$ labeled by $\boxed{1}$. Then either there is an edge $(0, a)$ or two edges $(0, i)$, $(i, a)$. Anyway, either $H_{\circledast}(V/W_{0,a}) = H$ or $H_{\circledast}(V/(W_{0,i} \oplus W_{i,a})) = H$. We apply the procedure of this lemma to $H$. Thus, arguing by induction, we construct a connected graph $\Gamma_q$ corresponding to some indecomposable triple (in general a subtriple of $(F, \check{F}, V)$). But, since $(F, \check{F}, V)$ is indecomposable, all direct factors of $F$ are vertices of $\Gamma_q$ and each $F$-invariant subspace of $V$ corresponds to an edge of $\Gamma_q$.

Assume that $\Gamma_q$ is not a tree. Let $\Gamma^c$ be the smallest connected subgraph of $\Gamma_q$ containing the root and the cycle. We may assume that either $\Gamma^c$ has two vertices and a double edge $(0, 1)$; or it has three vertices and a double edge $(1, 2)$. If this is not the case, we apply Lemma 4.2 and replace an edge by a vertex of weight 1. But neither $\mathrm{Sp}_1 \times \mathrm{Sp}_n : \mathbb{H}^n \oplus \mathbb{H}^n$ nor $\mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{Sp}_l : \mathbb{H}^n \oplus \mathbb{H}^n \otimes \mathbb{H}^l \oplus \mathbb{H}^n \otimes \mathbb{H}^l$ satisfies condition $(**)$. Thus we have shown that if a graph is not a tree, then the corresponding triple $(F, \check{F}, V)$ cannot satisfy condition $(**)$.

Assume that condition (I) is not fulfilled. Then there is a vertex $i$ such that $d(i) > 1$ which is connected with at least three vertices, say $j, t, s$. Assume that $j$ is not the root. Set $d := |d(i) - d(j)|$. Consider $F_*(W_{i,j}) : V/W_{i,j}$. Clearly, the corresponding graph contains a subgraph with a cycle, either



if $d(i) > d(j)$, or



if $d(i) \leq d(j)$. In the second case we have $d(i)$ vertices of weight 1.

Assume now that condition (II) is not fulfilled. Then we have edges $(i, j)$ and $(j, t)$ such that $d(i), d(j), d(t) > 1$. According to (I), the vertex $j$ has degree 2. If $i$ and $t$ are both of degree 1, then the triple $(F, \check{F}, V)$ is decomposable. Assume that $i$ has degree 2. Replace $F : V$ by $F_*(W_{j,t}) : V/W_{j,t}$, as shown on Picture 1. We will have at least $\min(d(j), d(t))$ new vertices connected with $i$. Thus in the new graph $i$ has degree at least 3, which is not allowed by condition (I).

Now we prove that if the triple $(F, \check{F}, V)$ corresponds to a tree $T_q$ described in the lemma, then it satisfies condition $(**)$. We argue by induction on the number of vertices. If $F = \mathrm{Sp}_1 \times \mathrm{Sp}_n$ we have nothing to prove (this was considered in Lemma 1.7). Moreover, if all vertices of $T_q$ are connected by edges with the root, then $(\mathrm{Sp}_1)_{\circledast}(V) = \mathrm{Sp}_1$ by Lemma 1.7. Take a vertex $i$ of degree 1, which is not connected by an edge with the root, and let $(i, j)$ be an edge of $T_q$. Clearly, $(F, \check{F}, V)$ satisfies condition $(**)$ if and only if $(F_*(W_{i,j}), \check{F}_*(W_{i,j}, V/W_{i,j})$ satisfies it.

Suppose $d(j) > 1$. Then $j$ has degree 2 and is connected by an edge with a vertex $s$ such that $d(s) = 1$ (we use conditions (I) and (II)). The edge $(i, j)$ is replaced by several new vertices, which are connected only with the vertex $s$ (see Picture 1). If $d(i) > d(j)$ and $d(i) < \infty$, then $F_*(W_{i,j})$ has a direct factor $\mathrm{Sp}_{d(j)-d(i)}$ acting trivially on $V/W_{i,j}$. It corresponds to an isolated vertex. Another connected component is a tree. If $d(i) \leq d(j)$ or $d(i) = \infty$, then the new graph is a tree. In all cases it satisfies conditions (I) and (II).

If $d(j) = 1$, then to obtain a new graph (or the connected component containing the root) we just erase the vertex $i$ and the edge $(i, j)$. $\square$

Suppose $T_q$ is a rooted tree satisfying conditions (I) and (II) of Lemma 4.1. Suppose $d(i) = 1$ and $i \neq 0$. Consider a path from 0 to $i$ and a direct sum $W(i)$ of all subspaces corresponding to the edges of this path.

$$1 = d(0) \;—\!—\; d(s) \;—\!—\; d(t) \;—\!—\; d(j) \;—\!—\; \cdots \;—\!—\; d(r) \;—\!—\; d(a) \;—\!—\; d(i) = 1$$

One can calculate, that $(\mathrm{Sp}_1 \times \mathrm{Sp}_{d(i)})_{\circledast}(W(i)) = \mathrm{Sp}_1$ is embedded diagonally in $\mathrm{Sp}_1 \times \mathrm{Sp}_{d(i)}$. Note that, each vertex $i$ with $d(i) = 1$ could be chosen as a root of $T_q$.

**Theorem 4.3.** *Let $X$ be a principal indecomposable commutative homogeneous space. Suppose $L_1 = \mathrm{Sp}_1$ is a simple direct factor of $L^\diamond$ and $L_1 \cap K = \pi_1(K) = \mathrm{U}_1$. Then $\mathfrak{n} = V$ is an Abelian Lie algebra, $L = F = L_1 \times \check{F}$, $K = \mathrm{U}_1 \times \check{F}$, and the triple $(F, \check{F}, V)$ corresponds to some rooted tree $T_q$ satisfying conditions (I) and (II) of Lemma 4.1.*

*Proof.* According to Lemma 1.14, $\pi_1(L_*) = \mathrm{Sp}_1$. Thus there is an $L$-invariant subspace $V \subset \mathfrak{n}$, such that the triple $(L^\diamond, L^\diamond/L_1, V)$ corresponds to a rooted tree $T_q$. In particular, each direct factor $L_j$ of $L^\diamond$ is $\mathrm{Sp}_n$. If $n > 1$, then $L_j \subset K$ according to Theorem 1.17 and Proposition 1.18. Suppose $L_2 = \mathrm{Sp}_1 \subset L^\diamond/\mathrm{Sp}_1$ is not contained in $K$. Then either $\pi_2(K) = \mathrm{U}_1$ or $\pi_2(K) = \mathrm{Sp}_1$. But $(L_1 \times L_2)_{\circledast}(\mathfrak{n}) \subset \mathrm{Sp}_1$, so if the first case takes place, then the equality $L = L_* K$ is impossible.

If $\pi_2(K) = \mathrm{Sp}_1$, then there is $L_3 \lhd L$ such that a direct factor $\mathrm{Sp}_1$ of $K$ is diagonally embedded in $L_2 \times L_3$.

$$\begin{tikzcd} L_3 \arrow[dr, dash] & \times & L_2 = \mathrm{Sp}_1 \arrow[dl, dash] & \mathrm{Sp}_1 \arrow[d, dash] \\ & \mathrm{Sp}_1 & & \mathrm{U}_1 \end{tikzcd}$$

Replacing $L$ by a smaller subgroup containing $K$, we may assume that $L_3 = \mathrm{Sp}_1$. Since $L = L_* K$, we have $(L_3 \times L_2 \times L_1)_{\circledast}(\mathfrak{n}) = \mathrm{Sp}_1 \times \mathrm{Sp}_1$. The projection of $K_*$ to $L_1 \times L_2 \times L_3$ is $\mathrm{U}_1$ which is not a spherical subgroup of $\mathrm{Sp}_1 \times \mathrm{Sp}_1$.

Thus $(L^\diamond/L_1) \subset K$. Since $X$ is principal, $U_1 \subset L_1$. Because $X$ is indecomposable, $P$ is trivial. To conclude, we show that $\mathfrak{n}$ is Abelian. Any irreducible $L$-invariant subspace $W_{i,j} \subset V$ is either $HS_0^2\mathbb{H}^n$ or $\mathbb{H}^n \otimes \mathbb{H}^m$. If $[W_{i,j}, W_{i,j}] \neq 0$, then, according to Table 3.1, $d(i) = 1$ and $[W_{i,j}, W_{i,j}] = \mathfrak{sp}_{d(i)}$. But $\mathfrak{sp}_{d(i)}$ is not contained in $V$. Hence, $[V, V] = 0$. Since $X$ is indecomposable, $\mathfrak{n} = V$.

We have not checked yet, that each space $(V \rightthreetimes F)/(\mathrm{U}_1 \times \check{F})$ is really commutative. Condition (A) of Theorem 1.3 is satisfied by construction of $T_q$. The Lie algebra $V$ is Abelian, thus condition (C) is also satisfied. Recall that $F_*(V) = \mathrm{Sp}_1 \times \check{F}_*(V)$, hence, $(\mathrm{U}_1 \times \check{F})_*(V) = \mathrm{U}_1 \times \check{F}_*(V)$. Thus $(\mathrm{U}_1 \times \check{F})_*(V)$ is a spherical subgroup of $F_*(V)$ and condition (B) holds. $\square$

We illustrate the structure of such a space by the following diagram.

$$\mathrm{U}_1 \longrightarrow\!\!\!\!\!\!- \mathrm{Sp}_1 \longrightarrow T_q$$

Here the direct factor $\mathrm{Sp}_1$ of $L$ corresponds to the root of $T_q$.

From now on we assume that $\pi_i(K) = L_i$ for each $L_i \cong \mathrm{Sp}_1 \subset L^\diamond$. Let $L^\triangle$ be a normal subgroup of $L$. Denote by $\pi^\triangle$ the projection $L \to L^\triangle$.

**Lemma 4.4.** *Let $L^\triangle = L_1{\times}L_2{\times}L_3{\times}L_4{\times}L_5$, where $L_i = \mathrm{Sp}_1$. Suppose $\pi^\triangle(K) = \mathrm{Sp}_1 \times \mathrm{Sp}_1$, where the first direct factor is diagonally embedded in $L_1{\times}L_2{\times}L_3$ and the secong in $L_4{\times}L_5$. Then $\pi^\triangle(L_*) = L_1{\times}L_2{\times}L_3 \times (L_4{\times}L_5)_{\circledast}(\mathfrak{n})$.*

*Proof.* We have $L^\triangle = \pi^\triangle(L_*)\pi^\triangle(K)$. Thus $(\mathrm{Sp}_1)^3 \subset \pi^\triangle(L_*)$. According to Lemma 1.14, $(L_1{\times}L_2{\times}L_3)_{\circledast}(\mathfrak{n}) = L_1{\times}L_2{\times}L_3$. If the statement of this lemma is not true, then there is a direct factor $\mathrm{Sp}_1$ of $\pi^\triangle(L_*)$, which has non-trivial projections on, say, $L_3$ and $L_4$. In particular, $\pi^\triangle(L_*) \subset (\mathrm{Sp}_1)^4$. There are three different possibilities for $\pi^\triangle(L_*)$, namely $(\mathrm{Sp}_1)^4$, $(\mathrm{Sp}_1)^3 \times \mathrm{U}_1$ and $(\mathrm{Sp}_1)^3$, but $\pi^\triangle(K_*)$ is never a spherical subgroup of $\pi^\triangle(L_*)$. □

Denote by $Fr_m$ the forest of $m$ rooted trees, satysfying conditions (I) and (II) of Lemma 4.1. We say that a triple $(F, \check{F}, V)$ corresponds to a forest $Fr_m$ if it is a product ($V$ is a direct sum) of $m$ triples $(F_i, \check{F}_i, V_i)$ corresponding to trees of this forest.

**Theorem 4.5.** *Let $X$ be an indecomposable commutative homogeneous space. Suppose $(L, K)$ contains a subpair $(L^\triangle, K^\triangle)$, where $L^\triangle = (\mathrm{Sp}_1)^3$, $K^\triangle = \mathrm{Sp}_1$, and $L^\triangle \subset L^\diamond$. Then there is a triple $(F, \check{F}, V)$ corresponding to a forest $Fr_3$ such that $L = F$, $K = \mathrm{Sp}_1 \times \check{F}$ and $\mathfrak{n} = V$ is an Abelian Lie algebra.*

*Proof.* Due to Lemma 1.14, $(L^\triangle)_{\circledast}(\mathfrak{n}) = L^\triangle$. Thus we can construct three differnt trees starting from direct factors of $L^\triangle$. These trees do not intersect, because otherwise $L^\triangle_{\circledast}(\mathfrak{n})$ would be a subgroup of $\mathrm{Sp}_1{\times}\mathrm{Sp}_1$.

The rest of the proof is similar to the proof of Theorem 4.3. Assume that there is a vertex $i$ of the first tree such that $d(i) = 1$ and $L_4 = \mathrm{Sp}_{d(i)}$ is not contained in $K$. Then there is $L_5 \lhd L$ such that a direct factor $\mathrm{Sp}_1$ of $K$ is diagonally embedded in $L_4 \times L_5$.

$$\begin{tikzcd}
L_5 \arrow[dr, dash] & \times & L_4 = \mathrm{Sp}_1 \arrow[dl, dash] & & \mathrm{Sp}_1 \arrow[drr, dash] & \times & \mathrm{Sp}_1 \arrow[d, dash] & \times & \mathrm{Sp}_1 \arrow[dll, dash] \\
& \mathrm{Sp}_1 & & & & & \mathrm{Sp}_1 & &
\end{tikzcd}$$

We may assume that $L_5 = \mathrm{Sp}_1$. Clearly, $(L_4{\times}L_1)_{\circledast}(\mathfrak{n}) = \mathrm{Sp}_1$ is embedded diagonaly into $L_4{\times}\mathrm{Sp}_{d(0)}$, where $Sp_{d(0)}$ is a direct factor of $L^\triangle$. This contradicts Lemma 4.4.

Thus $(L^\diamond/L^\triangle) \subset (K/K^\triangle)$. Because $X$ is indecomposable, $P$ is trivial. Applying the same argument as in Theorem 4.3, we get $[V, V] = 0$ and $\mathfrak{n} = V$. To conclude, we show that condition (B) of Theorem 1.3 holds. Here $(\mathrm{Sp}_1 \times \check{F})_*(V) = \mathrm{Sp}_1 \times \check{F}_*(V)$ is a spherical subgroup of $F_*(V) = (\mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_1)\check{F}_*(V)$. □

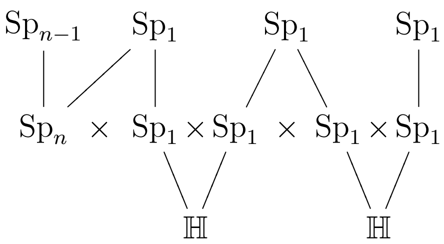
This space is shown on the second diagram in Theorem 4.11.

Note that the proof is valid in cases $L_1 \subset L^\diamond$, $L_2 \times L_3 \subset P$ (where we would have one non-trivial tree) and $L_1 \times L_2 \subset L^\diamond$, $L_3 \subset P$ (which corresponds to a forest $Fr_2$).

**Lemma 4.6.** *Let $L^\triangle = P_1 \times L_1 \times L_2 \times L_3$, where $L_i = \mathrm{Sp}_1$, $P_1 = \mathrm{SU}_n \subset P$. Suppose $\pi^\triangle(K) = (\mathrm{S})\mathrm{U}_{n-1} \times \mathrm{Sp}_1 \times \mathrm{Sp}_1$, where the last direct factor $\mathrm{Sp}_1$ of $\pi^\triangle(K)$ is diagonally embedded in $L_2 \times L_3$. Then $\pi^\triangle(L_*) = P_1 \times L_1 \times (L_2 \times L_3)_{\circledast}(\mathfrak{n})$.*

*Proof.* We pair $(\pi^\triangle(L_*), \pi^\triangle(K_*)$ is spherical and $P_1 \subset \pi^\triangle(L_*)$. Hence, $(L_1)_{\circledast}(\mathfrak{n}) = L_1$. If the statement of this lemma is not true, then $(L_1 \times L_2)_{\circledast}(\mathfrak{n}) = \mathrm{Sp}_1$ is embedded diagonally in $L_1 \times L_2$. Hence, $\pi^\triangle(L_*) = P_1 \times \mathrm{Sp}_1 \times (L_3)_{\circledast}(\mathfrak{n})$, $\pi^\triangle(K_*) = (\mathrm{S})\mathrm{U}_{n-1} \times (L_3)_{\circledast}(\mathfrak{n})$, and the pair $(\pi^\triangle(L_*), \pi^\triangle(K_*))$ cannot be spherical. $\square$

*Example* 11. Let $(L, K) = (\mathrm{Sp}_n \times \mathrm{Sp}_1, \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1) \times (\mathrm{Sp}_1 \times \mathrm{Sp}_1, \mathrm{Sp}_1) \times (\mathrm{Sp}_1, \mathrm{Sp}_1)$. Suppose $\mathfrak{n} = \mathbb{H} \oplus \mathbb{H}$ is an Abelian Lie algebra, $\mathrm{Sp}_n \subset P$, and the action $(L/P) : \mathfrak{n}$ is defined by the following diagram.



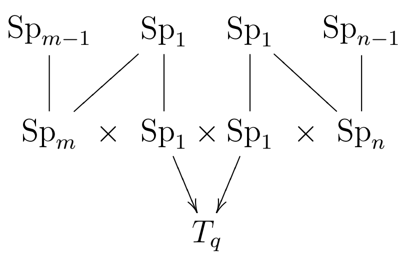

Then $X = (N \rtimes L)/K$ is commutative. Here $L_* = \mathrm{Sp}_n \times \mathrm{Sp}_1 \times \mathrm{Sp}_1$, $K_* = \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1$. The graph corresponding to $(L/P) : \mathfrak{n}$ is a forest with *two* (not four) trees.

$$1 \text{---} 1 \qquad\qquad 1 \text{---} 1$$

Consider a slight modification of this example.

*Example* 12. Let $(L, K) = (\mathrm{Sp}_n \times \mathrm{Sp}_1, \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1) \times (\mathrm{Sp}_1 \times \mathrm{Sp}_m, \mathrm{Sp}_1 \times \mathrm{Sp}_{m-1})$. Suppose $\mathfrak{n} = \mathbb{H}$ is an Abelian Lie algebra, $(\mathrm{Sp}_n \times \mathrm{Sp}_m) \subset P$, and each of $\mathrm{Sp}_1$ direct factors of $L$ acts on $\mathfrak{n}$ non-trivially. Then $X = (N \rtimes L)/K$ is commutative. Here $L_* = \mathrm{Sp}_n \times \mathrm{Sp}_1 \times \mathrm{Sp}_m$, $K_* = \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1 \times \mathrm{Sp}_{m-1}$. The graph corresponding to $(L/P) : \mathfrak{n}$ is a vertex, i.e., it is *one* tree instead of two.

We will see that we can insert any tree $T_q$ satisfying conditions (I), (II) of Lemma 4.1, instead of this vertex. The following diagram



illustrates that both direct factors $\mathrm{Sp}_1$ are vertices of weight 1 in one and the same tree.

Suppose we have a triple $(F, \check{F}, V)$, where $F = \mathrm{Sp}_1 \times \check{F}$ and $(\mathrm{Sp}_1)_{\circledast}(V) = \mathrm{U}_1$. Then we can construct a commutative homogeneous space $(V \leftthreetimes L)/K$ with $L = (\mathrm{Sp}_1 {\times} \mathrm{Sp}_1) \times \check{F}$, $K = \mathrm{Sp}_1 \times \check{F}$, where the direct factor $\mathrm{Sp}_1$ of $K$ is embedded diagonaly in $\mathrm{Sp}_1{\times}\mathrm{Sp}_1$. Thus, we have to describe all such triples $(F, \check{F}, V)$.

**Lemma 4.7.** *Suppose $F : V$ is an irreducible faithful representation of a connected compact group $F = \mathrm{Sp}_1{\cdot}\check{F}$ and $F_*(V) = \mathrm{U}_1{\cdot}\check{F}_*(V)$. Then there are three possibilities:*
*$\check{F} = \{e\}$, $V = \mathbb{R}^3$; $\check{F} = (\mathrm{S})U_n$, $V = \mathbb{C}^2{\otimes}\mathbb{C}^n$; $\check{F} = \mathrm{U}_1{\cdot}\mathrm{Sp}_n$, $V = \mathbb{C}^2{\otimes}\mathbb{C}^{2n}$.*

*Proof.* We have $V = V_1 \otimes_{\mathbb{D}} V^2$, where $\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $\mathrm{Sp}_1$ acts only on $V_1$ and $\check{F}$ only on $V^1$. Cases $\mathbb{D} = \mathbb{R}$ and $\mathbb{D} = \mathbb{C}$ were considered in Example 10. They yield exactly the three possibilities of this lemma.

Suppose $\mathbb{D} = \mathbb{H}$. Then the complexefication of $V_1 \otimes_{\mathbb{H}} V_2$ is an irreducible representation $W = V_1 \otimes_{\mathbb{C}} V_2$ of $F(\mathbb{C})$ such that $\check{F}(\mathbb{C}) \subset \mathrm{Sp}(V_2)$. Moreover, the stabiliser $(\mathrm{SL}_2)_x$ of a generic point $x \in \mathbb{P}V_1$ is infinite (here $\mathbb{P}V_1$ stands for the the projectification of $V_1$). Hence, $V_1 = \mathbb{C}^2$.

If $\dim V^2 = 1$, then $F_*(V) = \check{F} = \mathrm{U}_1$. But the representation $(\mathrm{Sp}_1{\cdot}\mathrm{U}_1) : \mathbb{H}$ is reducible. Thus $\dim V^2 > 1$. Since $V_2$ is an irreducible symplectic representation of $\check{F}(\mathbb{C})$, the group $\check{F}$ is semisimple. Now we use the second classification of Elashvili [15]. We represantation $\check{F}(\mathbb{C}) : V_2$ cannot have a tensor "factor" $\mathrm{SL}_m : \mathbb{C}^m$ with $m > 2$. Hence, according to [15, §3], the pair $(F(\mathbb{C}), \mathbb{C}^2 \otimes V_2)$ is contained in Tables 5 and 6 of [15]. Note that there are a few inaccuracies in these tables, which are corrected in [39].

Anyway, we have three possibilities: $\check{F}(\mathbb{C}) = \mathrm{SL}_2 \times \mathrm{SL}_4$, $V_2 = \mathbb{C}^2 \otimes \mathbb{C}^4$; $\check{F}(\mathbb{C}) = \mathrm{SL}_6$, $V_2 = \bigwedge^3 \mathbb{C}^6$ and $\check{F}(\mathbb{C}) = \mathrm{SO}_{12}$, $V_2$ being a "half-spinor" representation. But in all three cases a generic stabiliser $(F(\mathbb{C}))_*(W)$ is contained in $\check{F}(\mathbb{C})$. Thus, if $\mathbb{D} = \mathbb{H}$, $\dim V > 1$ and $V$ is irreducible, then either $F_*(V) = \check{F}_*(V)$ or $F_*(V) = \mathrm{Sp}_1{\cdot}\check{F}_*(V)$. □

**Lemma 4.8.** *Let $\Gamma_q$ be a connected weighted graph with $q{+}1$ vertices. Suppose $F = \prod\limits_i \mathrm{Sp}_{d(i)}$, $V = \bigoplus\limits_{(i,j)} W_{i,j}$, where $W_{i,j} = \mathbb{H}^{d(i)}{\otimes}\mathbb{H}^{d(j)}$. If $\Gamma_q$ contains two different minimal cycles, then $(\mathrm{Sp}_{d(i)})_{\circledast}(V)$ is finite for each $i$ such that $d(i) = 1$.*

*Proof.* Suppose $(\mathrm{Sp}_{d(i)})_{\circledast}(V)$ is either $\mathrm{Sp}_1$ or $\mathrm{U}_1$. The statement of Lemma 4.2 is true in this more general situation. Thus, we can pass to $\widetilde{\Gamma}_{q-1}$. In other words, we may assume that $q \le 3$ and that $\Gamma_q$ contains two double edges or one triple edge. In each case one can verify that $(\mathrm{Sp}_1)_{\circledast}(V)$ is finite. □

Let $\Gamma_q$ be a weighted graph with several *special* vertices $a_1, \ldots, a_k$ such that $d(a_j) = 1$. Suppose that $\Gamma_q$ contains no triple edges, each infinite vertex has degree 1 and if $(i, j)$ is an infinite edge with $j = \infty$, then $1 < d(i) < \infty$ and there is at most one infinity vertex $t \ne j$ such that $(i, t)$ is an edge of $\Gamma_q$. We construct a compact group $H = H(\Gamma_q)$, a vector space

$V = V(\Gamma_q)$, and a linear action $H : V$ by the following principle. For each non-special finite vertex $i$ we set $H(i) = \mathrm{Sp}_{d(i)}$. To special vertex $j$ we attache a group $H(j)$ and a vector space $\check{V}(j)$. There are several possibilities: $H(j) = \mathrm{U}_1$, $\check{V}(j)$ is a zero-dimensional vector space; $H(j) = \mathrm{Sp}_1$, $\check{V}(j) = \mathbb{R}^3$; $H(j) = \mathrm{Sp}_1{\times}(\mathrm{S})\mathrm{U}_4$, $V(j)$ is either $\mathbb{C}^2{\otimes}\mathbb{C}^4$ or $\mathbb{C}^2{\otimes}\mathbb{C}^4 \oplus \mathbb{R}^6$; $H(j) = \mathrm{Sp}_1{\times}(\mathrm{S})\mathrm{U}_{m(j)}$ where $m(j) \geq 1$, $m(j) \neq 2, 4$, $\check{V}(j) = \mathbb{C}^2{\otimes}\mathbb{C}^{m(j)}$. Set $\hat{H} := \prod_i H(i)$ and $H := \hat{H} \times (\mathrm{U}_1)^r \times (\mathrm{U}_1)^s$, where $r$ is the number of double edges, $s$ is the number of pairs of infinite edges $(i, j)$, $(i, t)$ with $d(j) = d(t) = \infty$ and $j \neq t$; $V := \left(\bigoplus_{(i,j)} W_{i,j}\right) \oplus \bigoplus_{\text{special } j} \check{V}(j)$, where $W_{i,j}$ are defined in the same way as for the tree $T_q$. Each group $H(i)$ acts on $W_{i,j}$, also for each special vertex $j$ the group $H(j)$ naturally acts on $\check{V}(j)$. Thus the actions $H(i) : V$ are well-defined. Each $\mathrm{U}_1 \subset (\mathrm{U}_1)^r$ acts on a subspace $\mathbb{H}^{d(i)}{\otimes}\mathbb{H}^{d(t)}{\oplus}\mathbb{H}^{d(i)}{\otimes}\mathbb{H}^{d(t)}$ corresponding to a finite double edge $(i, t)$, and each $\mathrm{U}_1 \subset (\mathrm{U}_1)^s$ acts on a subspace $HS^2_0\mathbb{H}^{d(i)} \oplus HS^2_0\mathbb{H}^{d(i)}$, corresponding to infinity edges $(i, j)$, $(i, t)$.

We call a triple $(F, \check{F}, V)$ *principal* if the action $F : V$ satisfies the second condition of Definition 6.

**Lemma 4.9.** *Each principal indecomposable triple $(F, \check{F}, V)$ such that $\mathfrak{f} \subset \mathfrak{so}(V)$, $F = \mathrm{Sp}_1{\times}\check{F}$, and $(\mathrm{Sp}_1)_{\circledast}(V) = \mathrm{U}_1$ corresponds to a described above graph $\Gamma_q$ in a sense that $\hat{H} \subset F \subset H(\Gamma_q)$ and $\mathrm{Sp}_1$ is a subgroup attached to a vertex $i$ of $\Gamma_q$ with $d(i) = 1$. Conversely, suppose a triple $(F, \check{F}, V)$ corresponds to a graph $\Gamma_q$. Then $(F, \check{F}, V)$ is principal and $(\mathrm{Sp}_1)_{\circledast}(V) = \mathrm{U}_1$ if and only if one of the following four possibilities takes place.*

*(1) $\Gamma_q$ is a tree $T_q$ satisfying conditions (I), (II) of Lemma 4.1 with one special vertex $j$, and if $H(j) = \mathrm{U}_1$, then $j$ has degree 1.*

*(2) $\Gamma_q$ satisfies conditions (I), (II), contains one minimal cycle and no special vertices.*

*(3) $\Gamma_q$ is a tree, it contains no special vertices, satisfies condition (II), there is only one vertex $i$ of degree 3 such that $d(i) > 1$, all other vertices with $d(j) > 1$ have degrees at most 2, if $d(i) > 2$ and $i$ has degree three, then $d(t) = 1$ for each edge $(i, t)$.*

*(4) $\Gamma_q$ is a tree, it contains no special vertices, satisfies condition (I), there is only one edge $(i, j)$ such that $d(i), d(j) > 1$ and both vertices $i$ and $j$ have degree 2, for that edge $d(i) = 2$ and if $d(j) > 2$, then $d(t) = 1$ for the unique edge $(j, t)$ such that $t \neq i$.*

*Proof.* Suppose we have such a triple $(F, \check{F}, V)$. A graph $\Gamma_q$ can be constructed by the same procedure as in Lemma 4.1. Again we start with the vertex 0 (the root) of weight $d(0) = 1$. If $V = W_1{\oplus}W_2$ and $(\mathrm{Sp}_1)_{\circledast}(W_1) = (\mathrm{Sp}_1)_{\circledast}(W_2) = \mathrm{U}_1$, then $(\mathrm{Sp}_1)_{\circledast}(V)$ is finite. Hence, there is at most one irreducible $F$-invariant subspace $W \subset V$ such that $(\mathrm{Sp}_1)_{\circledast}(W) = \mathrm{U}_1$.

Suppose such $W$ exists. Then 0 is a special vertex. Let $\mathrm{Sp}_1{\times}H \subset F$ be the maximal connected subgroup acting on $W$ locally effectively. Then $(\mathrm{Sp}_1{\times}H) : W$ satisfies conditions of Lemma 4.7. Moreover, $(\mathrm{Sp}_1{\times}H_{\circledast}(V/W)) : W$ also satisfies those conditions. It can be easily seen, that either $H = \mathrm{U}_2$, $W = \mathbb{H}^1 \otimes_{\mathbb{C}} \mathbb{C}^2$ or $\mathrm{Sp}_1{\times}H$ is one of the groups that can be attached to a special vertex. In case $H = \mathrm{U}_2$ we put a vertex $i$ with $d(i) = 1$ into $\Gamma_q$ and a double edge $(0, i)$. The centre of $\mathrm{U}_2$ will be a direct factor in $(\mathrm{U}_1)^r \subset H(\Gamma_q)$. Consider the second case, where $\mathrm{Sp}_1 \times H = H(0)$. If $H \neq \mathrm{SU}_4$, then the action $H : (V/W)$ is trivial and we set $\check{V}(0) = W$. In case $H = \mathrm{SU}_4$, either $\check{V}(0) = W \oplus \mathbb{R}^6$ or $\check{V}(0) = W$. Clearly $H$ acts on $W/\check{V}(0)$ trivially, and the triple $(F/H, \check{F}/H, V/W)$ satisfies condition $(**)$. Thus we can construct a tree $T_q$ by Lemma 4.1.

Suppose now the root is not a special vertex. Then each irreducible $F$-invariant subspace $W \subset V$ on which $\mathrm{Sp}_1$ acts non-trivially is of the form $\mathbb{H}^1 \otimes \mathbb{H}^n$ and we construct the first level of the graph. Unlike the situation of Lemma 4.1, it can contain double edges. Also if $n = 1$, then $F$ can act on $\mathbb{H}^1 \otimes \mathbb{H}^n$ as $\mathrm{Sp}_1 \times \mathrm{U}_1$. Nevertheless, we proceed in the same manner as in Lemma 4.1 and construct a graph $\Gamma_q$.

Now we prove that $\Gamma_q$ belongs to one of the four types listed in this lemma. Suppose the root is a special vertex, then $V = \check{V}(0) \oplus \widetilde{V}$, where $\widetilde{V}$ is an $F$-invariant complement, and $(\mathrm{Sp}_1)_{\circledast}(\check{V}(0)) = \mathrm{U}_1$. Thus, $(\mathrm{Sp}_1)_{\circledast}(\widetilde{V}) = \mathrm{Sp}_1$ and $\Gamma_q$ is a tree $T_q$ by Lemma 4.1. Assume that the root is not a special vertex.

For each special vertex $j$, we can replace $H(j)$ by $H(j)_*(\check{V}(j)) = \mathrm{U}_1$ and $\check{V}(j)$ by a zero-dimensional vector space without any alteration of $(\mathrm{Sp}_1)_{\circledast}(V)$. On the other hand, suppose we have a double edge $(j, t)$ such that $d(j) = d(t) = 1$, $H(j) = H(t) = \mathrm{Sp}_1$ and $t$ has degree 2. Then we can erase this double edge and replace $H(i)$ by $H(i)_{\circledast}(2\mathbb{H}) = \mathrm{U}_1$. Thus, we may insert instead of each special vertex $j$ a double edge $(j, t)$ where $H(j) = H(t) = \mathrm{Sp}_1$ and $t$ is a new vertex of degree 2.

Recall that infinite vertices corresponds to irreducible $F$-invariant subspaces $HS_0^2\mathbb{H}^n \subset V$. In particular, if $d(i) = \infty$, then $i$ has degree 1 and is not contained in any minimal cycle of $\Gamma_q$. Thus we can apply Lemma 4.8 to $\Gamma_q$. It follows that $\Gamma_q$ contains at most one minimal cycle and if it contains a special vertex, then it is a tree and has no other special vertices.

If condition (I) or (II) is not satisfied, we erase one edge $(i, j)$ and replace $H$ by $H_*(W_{i,j})$. According to Picture 1, the graph corresponding to $H_*(W_{i,j}) : V/W_{i,j}$ contains a cycle. Thus, there is only one place in $\Gamma_q$ in which one of this conditions is not satisfied. Other possibilities yield a contradiction by the same argument. Note that, since $F : V$ is principal, if $H(j) = \mathrm{U}_1$, then $H(j)$ acts non-trivially on the only one irreducible $F$-invariant subspace and $j$ has degree 1.

One can prove by induction that if $\Gamma_q$ is one of the four graphs described in this lemma, then $(\mathrm{Sp}_1)_{\circledast}(V) = \mathrm{U}_1$ (see the last part of the proof of Lemma 4.1). □

In the following $T_q$ stands for a tree satisfying conditions (I) and (II) of Lemma 4.1, $\Gamma_q$ for a graph of one of the four types described in Lemma 4.9.

**Lemma 4.10.** *Suppose a triple $(F, \check{F}, V)$ corresponds to $\Gamma_q$. Let $\mathfrak{n}$ be a non-Abelin Lie algebra such that $V \subset \mathfrak{n}$, $\mathfrak{n}$ is generated by $V$, and $\mathfrak{n} = V \oplus \mathbb{R}^t$, where $\mathbb{R}^t$ is a trivial $F$-module. If the action $F : \mathfrak{n}$ is commutative then either*

- *$\Gamma_q$ contains a double edge $(i, j)$, $F = \hat{H}(\Gamma_q) \times \mathrm{U}_1$, and $\mathfrak{n}' = [2W_{i,j}, 2W_{i,j}] = \mathbb{R}$;*

- *or $\Gamma_q$ contains a special vertex $j$ and the Lie algebra structure on $\mathfrak{n}$ depends on $H(j)$ in the following way:*
  - *if $H(j) = \mathrm{U}_1$, then there is a single vertex $i$ connected with $j$ by an edge and $\mathfrak{n}' = [W_{j,i}, W_{j,i}] = \mathbb{R}$;*
  - *if $H(j) = \mathrm{Sp}_1{\times}(\mathrm{S})\mathrm{U}_m$, then $\mathfrak{n}' = [\hat{V}(j), \hat{V}(j)] = \mathbb{R}$;*
  - *if $H(j) = \mathrm{Sp}_1$, then $\hat{V}(j) = \mathbb{R}^3 \cong \mathfrak{sp}_1$ and $\mathfrak{n} = V$ with $[W_{j,i}, W_{j,i}] = \hat{V}(j)$ for some edges $(j,i)$.*

*Proof.* Assume $\Gamma_q$ contains no special vertices and no double edges. Then each irreducible $F$-invariant subspace $W \subset V$ is either $W_{i,j}$ or $HS_0^2\mathbb{H}^{d(i)}\otimes_{\mathbb{C}}\mathbb{C}$. According to Table 3.1, $[W, W]$ could be non-trivial only if $W = \mathbb{H}\otimes\mathbb{H}^n$. In that case $[W, W] = \mathbb{H}_0$. But the action $F : \mathbb{H}_0$ is non-trivial and $\mathbb{H}_0$ is not a subspace of $V$. A contradiction. By the same reason, if neither $j$ nor $i$ is special and $(i,j)$ is not a double edge, then $[W_{i,j}, W_{i,j}] = 0$.

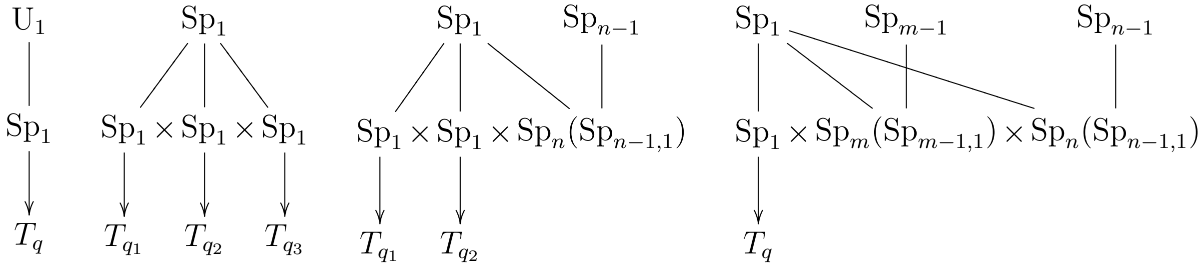
Suppose $\Gamma_q$ contains a double edge $(i,j)$. Then both vertices $i$ and $j$ has degree at least two. Hence by condition (II) one of them, assume that $i$, has weight 1. According to Table 3.1, the action $(\mathrm{U}_2{\times}\mathrm{Sp}_n) : \mathbb{C}^2\otimes\mathbb{H}^n \oplus \mathbb{R}$ is commutative, but without the central subgroup $\mathrm{U}_1$, it is not.

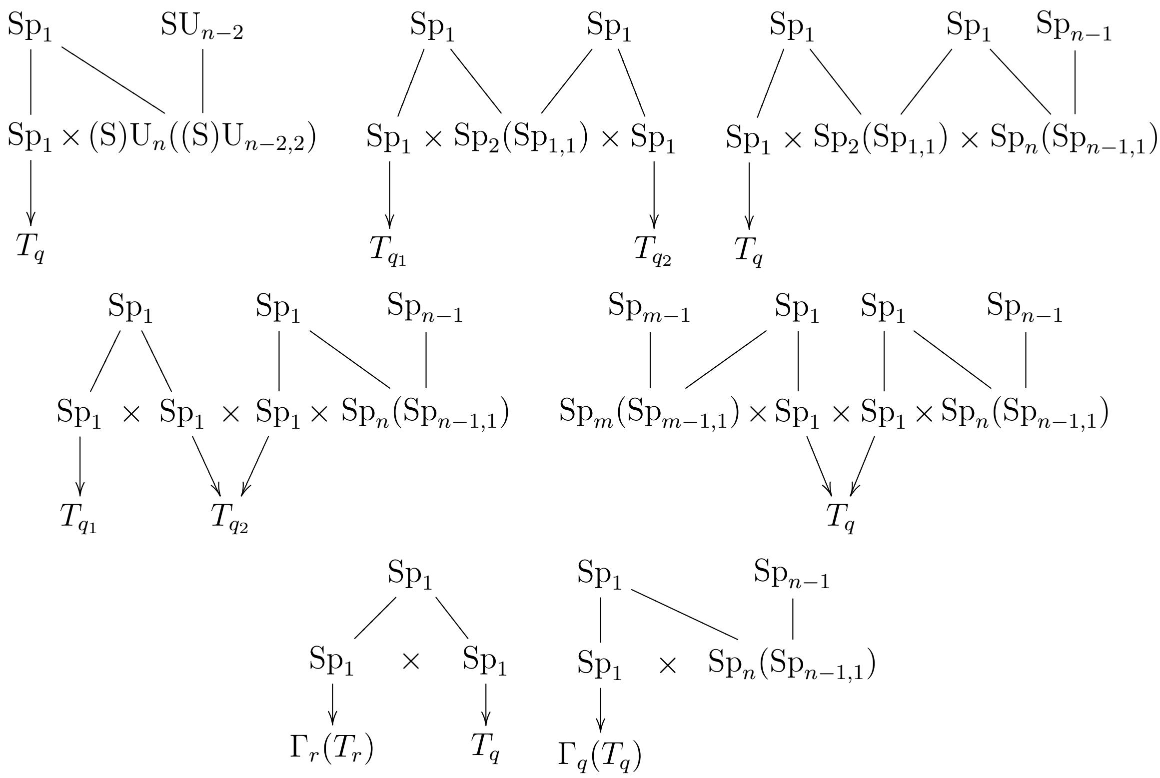
Now let $j$ be a special vertex. If $H(j) = \mathrm{U}_1$, then $j$ has degree 1. Let $i$ be the unique vertex connected with $j$ by an edge. According to Table 3.1, $[W_{i,j}, W_{i,j}] \subset HS_0^2\mathbb{H}^{d(i)} \oplus \mathbb{H}_0$, where $\mathbb{H}_0 = \mathfrak{u}_1 \oplus \mathbb{R}$ as an $\mathrm{U}_1$-module. Clearly, $\mathfrak{u}_1$ is not a subspace of $V$. In case $d(i) = 1$, the space $HS_0^2\mathbb{H}^{d(i)}$ is zero-dimensional. If $d(i) > 1$, then $i$ has degree 2 and there is no edges $(s,i)$ with $d(s) = \infty$. Thus, in this case $HS_0^2\mathbb{H}^{d(i)}$ is not a subspace of $V$ and $[W_{i,j}, W_{i,j}] = \mathbb{R}$.

If $H(j) = \mathrm{Sp}_1{\times}(\mathrm{S})\mathrm{U}_m$, then we again use Table 3.1, to check that $[\check{V}(j), \check{V}(j)] = \mathbb{R}$. Consider the last case $H(j) = \mathrm{Sp}_1$. Here $[W_{i,j}, W_{i,j}]$ can be either $\check{V}(j)$ or zero for each edge $(i,j)$. Since all actions $\mathrm{Sp}_{d(i)} : W_{i,j}\oplus\mathbb{R}^3$ are commutative, $F : \mathfrak{n}$ is also commutative. $\square$

In the following, the attache to a graph $\Gamma_q$ not only a triple $(F, \check{F}, V)$, but also a Lie algebra $\mathfrak{n} = \mathfrak{n}(\Gamma_q)$ be the principle of Lemma 4.10. A diagram $\mathrm{Sp}_1 \longrightarrow T_q(\Gamma_q)$ means that $\mathrm{Sp}_1$ corresponds to a vertex of weight 1 of $T_q$ or $\Gamma_q$.

**Theorem 4.11.** *Let $X$ be a maximal principal indecomposable commutative homogeneous space of non-reductive group $G$. Suppose $X$ is not $\mathrm{Sp}_1$-saturated and is not of Hiesenberg type. Then it belongs to one of the following 11 clases.*






*Proof.* Suppose there is a direct factor $L_i \subset L^\diamond$ such that $\pi_i(K) \neq L_i$. If $L_i \neq \mathrm{Sp}_1$, then $X$ is contained in Table 1.2b due to Proposition 1.18 and $X$ is $\mathrm{Sp}_1$-saturated. Hence, $\mathrm{L}_i = \mathrm{Sp}_1$ and $L_i \cap K = \mathrm{U}_1$. By Theorem 4.3, $X$ corresponds to a tree $T_q$ satisfying conditions (I) and (II). This space is shown on the first diagram. From now on assume that $\pi_i(K) = L_i$ for each simple direct factor $L_i \lhd L^\diamond$.

Suppose $P$ is non-trivial. Then there is a simple direct factor $K_1 \subset K$, which is contained in neither $P$ nor $L^\diamond$. If $K_1 \neq \mathrm{Sp}_1$, then by Theorem 1.17, $X$ is $((\mathbb{R}^n \rtimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$ or $((H_n \rtimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$. But both these spaces are $\mathrm{Sp}_1$-saturated. Thus $K_1 = \mathrm{Sp}_1$, and each simple normal subgroup $L_i \lhd L^\diamond$ such that $L_i \neq \mathrm{Sp}_1$ is contained in $K$.

Note that if $L_1 \subset P$ and $\pi_1(K) \neq L_1$, then we can replace $L_1$ by another real form of $L_1(\mathbb{C})$. For example, if $L_1 = \mathrm{Sp}_n$ and $\pi_1(K) = \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1$, then $L_1$ can also be $\mathrm{Sp}_{n-1,1}$. In calculations we assume $P$ to be compact.

Let $(L^\triangle, K^\triangle)$ be an indecomposable spherical subpair of $(L, K)$ such that $L^\triangle \neq K^\triangle$. Because $X$ is indecomposable, $L^\triangle$ is not contained in $P$, i.e., there is $L_i \subset (L^\triangle \cap L^\diamond)$. Since $L_i \not\subset K$, we have $L_i = \mathrm{Sp}_1$. According to Theorem 1.3, $L^\triangle = (L^\triangle)_\circledast(\mathfrak{n})K^\triangle$ and the pair $((L^\triangle)_\circledast(\mathfrak{n}), (K^\triangle)_\circledast(\mathfrak{n}))$ is spherical. There are only one spherical pair $(L^\triangle, K^\triangle)$ such that $(L^\triangle)_\circledast(\mathfrak{n})$ can be a proper subgroup of $L^\triangle$, namely $(\mathrm{Sp}_n \times \mathrm{Sp}_1, \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1)$. In that case $(L^\triangle)_\circledast(\mathfrak{n})$ can be $\mathrm{Sp}_n \times \mathrm{U}_1$. It follows that $(L_i)_\circledast(\mathfrak{n})$ is either $\mathrm{Sp}_1$ or $\mathrm{U}_1$ for each $L_i = \mathrm{Sp}_1 \lhd L$, $L_i \not\subset K$. Starting from each $L_i = \mathrm{Sp}_1 \subset L^\diamond$, which is not contained in $K$, we construct either a tree $T_q = T_q(L_i)$ or $\Gamma_q = \Gamma_q(L_i)$. If two different trees (or graphs) have a non-trivial intersection, then they coincide and $(L_i \times L_j)_{\circledast}(\mathfrak{n})$ is either $\mathrm{Sp}_1$ or $\mathrm{U}_1$. The second case is never possible in view of conditions (A) and (B) of Theorem 1.3.

In cases $L^{\triangle} = L_1 \times \mathrm{Sp}_m \times \mathrm{Sp}_n$, $L^{\triangle} = L_1 \times (\mathrm{S})\mathrm{U_n}$, and $L^{\triangle} = L_1 \times \mathrm{Sp}_2 \times \mathrm{Sp}_n$, where $L_1 = \mathrm{Sp}_1$ all subgroups corresponding to the vertices of $T_q(L_1)$ are contained in $K$. It can be proved by the same arguments as Theorem 4.5 and Lemma 4.6. These spaces are shown on the last 3 diagrams of the first row and in the second row. For each item one can directly verify that conditions (A) and (B) of Theorem 1.3 are satisfied. In all these cases $\mathfrak{n}$ is Abelian by the same reason as in Theorems 4.3, 4.5.

From now assume that $(L, K)$ is a product of pairs $(\mathrm{Sp}_n \times \mathrm{Sp}_1, \mathrm{Sp}_{n-1} \times \mathrm{Sp}_1)$ and a pair $(K^1, K^1)$, where $K^1$ is a compact Lie group. It can happen that $(\mathrm{Sp}_n{\times}\mathrm{Sp}_1{\times}\mathrm{Sp}_1{\times}\mathrm{Sp}_m)_{\circledast}(\mathfrak{n}) = \mathrm{Sp}_n{\times}\mathrm{Sp}_1{\times}\mathrm{Sp}_m$, where $\mathrm{Sp}_1$ is the diagonal of $\mathrm{Sp}_1{\times}\mathrm{Sp}_1$. In this case both subgroups $\mathrm{Sp}_1$ of $L$ are vertices of one and the same tree $T_q$. One can show that for all other vertices of $T_q$ groups $\mathrm{Sp}_{d(i)}$ are contained in $K$. If $n = 1$ or $m = 1$ or both, we can construct trees corresponding to these factors. They do not intersect, otherwise $(\mathrm{Sp}_n \times \mathrm{Sp}_1 \times \mathrm{Sp}_1 \times \mathrm{Sp}_m)_{\circledast}(\mathfrak{n})$ would be smaller. Here $\mathfrak{n} = V$ is also Abelian. As an example, we check that conditions (A) and (B) hold for the second space in the third row. Here the product of $\mathrm{Sp}_{d(i)}$ over all finite vertices of $T_q$ equals $(\mathrm{Sp}_1{\times}F_0{\times}\mathrm{Sp}_1)$ and $L = \mathrm{Sp}_m{\times}\mathrm{Sp}_1{\times}F_0{\times}\mathrm{Sp}_1{\times}\mathrm{Sp}_n$. We have $L_* = \mathrm{Sp}_m{\times}\mathrm{Sp}_1{\times}(F_0)_*(V){\times}\mathrm{Sp}_n$, $K_* = \mathrm{Sp}_{m-1}{\times}\mathrm{Sp}_1{\times}(F_0)_*(V){\times}\mathrm{Sp}_{n-1}$. Clearly, $L = L_*K$ and the pair $(L_*, K_*)$ is spherical (see item 3 of Table 2.3).

Finally, it is possible that $(\mathrm{Sp}_n{\times}\mathrm{Sp}_1)_{\circledast}(\mathfrak{n}) = \mathrm{Sp}_n{\times}\mathrm{U}_1$. Then we construct a graph $\Gamma_q$ starting with $\mathrm{Sp}_1$. It is enough to show that these graphs do not intersect for $\mathrm{Sp}_1$ from two different pairs of the type $(\mathrm{Sp}_1{\times}\mathrm{Sp}_1, \mathrm{Sp}_1)$. If two direct factors $L_1 \cong L_3 \cong \mathrm{Sp}_1$ corresponds to one and the same $\Gamma_q$, then $((\mathrm{Sp}_1)^4)_{\circledast}(\mathfrak{n}) \subset \mathrm{Sp}_1{\times}\mathrm{U}_1{\times}\mathrm{Sp}_1$. Due to condition (A) of Theorem 1.3 this group contains $\mathrm{Sp}_1{\times}\mathrm{Sp}_1$. But then for the subgroup $(\mathrm{Sp}_1{\times}\mathrm{Sp}_1) \lhd K$ we have $(\mathrm{Sp}_1{\times}\mathrm{Sp}_1)_{\circledast}(\mathfrak{n}) = \mathrm{U}_1$ and $\mathrm{U}_1$ is not a spherical subgroup of $\mathrm{Sp}_1{\times}\mathrm{Sp}_1$.

According to Lemma 4.10, in the last two cases $\mathfrak{n}$ can be non-Abelin, and we have to check condition (C) of Theorem 1.3. Recall that $\mathfrak{m} = \mathfrak{l}/\mathfrak{k}$. Let $(i, j)$ be a double edge of $\Gamma_q$ with $d(i) = 1$. If $d(j) > 1$, then $j$ has degree 2, i.e., $\mathrm{Sp}_{d(j)}$ acts on $V/2W_{i,j}$ trivially. Thus $K_*(\mathfrak{m}{\oplus}(V/2W_{i,j}))$ acts on $2W_{i,j} = \mathbb{C}^2{\otimes}\mathbb{H}^{d(j)}$ as $\mathrm{U}_1{\times}\mathrm{U}_1{\times}\mathrm{Sp}_{d(j)}$. The action $(\mathrm{U}_1{\times}\mathrm{U}_1{\times}\mathrm{Sp}_{d(j)}) : 2W_{i,j} \oplus \mathbb{R}$ is item 8 of Table 3.2. Hence, it is commutative. Suppose now $d(j) = 1$. If we erase the double edge $(i, j)$, then either $i$ or $j$ is not connected with the root (it follows from the fact that $\Gamma_q$ contains only one minimal cycle). Hence, in this case $K_*(\mathfrak{m} \oplus (V/2W_{i,j}))$ also acts on $\mathbb{C}^2{\otimes}\mathbb{H}^1$ as $\mathrm{U}_1{\times}\mathrm{U}_1{\times}\mathrm{Sp}_{d(j)}$.

Let $j$ be a special vertex of $\Gamma_q$. Suppose $H(j) = \mathrm{U}_1$. Then $j$ has degree 1 and there is the unique edge $(j, i)$. According to Lemma 4.10, $W_{i,j}$ is the only non-Abelian subspace of $\mathfrak{n}$ and $[W_{i,j}, W_{i,j}] = \mathbb{R}$. If $d(i) = 1$, $K_*(\mathfrak{m} \oplus (V/W_{i,j}))$ acts on $W_{i,j} \oplus \mathbb{R} \cong \mathbb{H} \oplus \mathbb{R}$ as $\mathrm{U}_1 \times \mathrm{U}_1$. This action is commutative. Suppose $d(i) > 1$, then $i$ has degree 2. Since $i$ is connected with the root, there is an edge $(i, t)$ with $t \neq j$ such that $t$ has degree at least 2. According to condition (II) of Lemma 4.1 $d(t) = 1$. We calculate that $K_*(\mathfrak{m} \oplus (V/W_{i,j}))$ acts on $W_{i,j} \oplus \mathbb{R}$

as $U_1{\times}U_1{\times}Sp_{d(i)-1}$. This action is also commutative.

Suppose now $H(j) = Sp_1{\times}(S)U_m$. Here $\hat{V}(j)$ is the only non-Abelian subspace of $\mathfrak{n}$ and $K_*(\mathfrak{m}{\oplus}(V/\check{V}(j)))$ acts on $\check{V}(j)$ as $U_1{\times}(S)U_m$. Since $m \neq 2$ the action $(U_1{\times}(S)U_m) : \hat{V}(j){\oplus}\mathbb{R}$ is commutative.

Finally, suppose that $H(j) = Sp_1$ and $\check{V}(j) = \mathbb{R}^3 \cong \mathfrak{sp}_1$. For each edge $(i,j)$ the stabiliser $K_*(\mathfrak{m} \oplus (V/W_{i,j}) \oplus \check{V}(j))$ acts on $W_{i,j}$ and on $W_{i,j} \oplus \hat{V}(j)$ as a product $Sp_{n_1}{\times} \ldots {\times}Sp_{n_r}{\times}Sp_1$, where $\sum n_s = d(i)$. This action is commutative. □

Now we consider non-$Sp_1$-saturated commutative spaces of Heisenberg type.

*Example* 13. Suppose $K_1 = SO_3$, $\mathfrak{n}_1 = \mathbb{R}^3 \oplus \mathfrak{sp}_1$ and a triple $(F, \check{F}, V)$ corresponds to a tree $T_q$ with $Sp_{d(0)}/\{\pm e\} = K_1$. Set $K = SO_3 \times \check{F}$, $\mathfrak{n} = \mathfrak{n}_1 \oplus V$. The Lie algebra structure on $\mathfrak{n}$ is given by the formulas $[W_{0,i}, W_{0,i}] = \mathfrak{sp}_1$ for each edge $(0,i)$, $[W_{i,j}, W_{i,j}] = 0$ for all other edges. Then $X = (N \leftthreetimes K)/K$ is commutative and maximal. Indeed, $(K_1)_{\circledast}(V) = K_1$ and for each $W_{0,i}$ we have $Sp_{n_1} \times \cdots \times Sp_{n_r} \subset K_*(\mathbb{R}^3 \oplus (V/W_{0,i}))$, where $\sum n_s = d(i)$, hence the action $K_*(\mathbb{R}^3 \oplus (V/W_{0,i})) : W_{0,i} \oplus [W_{0,i}, W_{0,i}]$ is commutative. We illustrate the structure of $X$ by the following diagram.

$$\begin{array}{c} SO_3 \ : \ \mathbb{R}^3 \oplus \mathfrak{sp}_1 \\ \downarrow \\ T_q \end{array}$$

Let $X_0 = (N_0 \leftthreetimes K_0)/K_0$ be an $Sp_1$-saturated commutative space of Heisenberg type with $\mathfrak{n}_0' \neq 0$. Suppose $K_0 = Z(L){\times}L_1{\times}\cdots{\times}L_s{\times}L_{s+1}{\times}\cdots{\times}L_m$, where $L_1 = L_2 = \cdots = L_s = Sp_1$ and a triple $(F, \check{F}, V)$ corresponds to $Fr_s$. We assume that each $L_i$ corresponds to the root of the $i$-th tree of $Fr_s$. Set $K = Z(L) \times L_{s+1} \times \cdots \times L_m \times (Sp_1)^s \times \check{F}$, $\mathfrak{n} = \mathfrak{n}_1 \oplus V$, where for each edge $(j,t)$ of the $i$-th tree $[W_{j,t}, W_{j,t}]$ is non-zero only if $j = 0$ and $\mathfrak{l}_i \subset \mathfrak{n}_0'$, in that case $[W_{0,t}, W_{0,t}]$ can be $\mathfrak{l}_i$. We do not require that $[W_{0,t}, W_{0,t}] = \mathfrak{l}_i$ for all edges $(0,t)$ of the $i$-th tree. By the same argument as in Example 13, one can show that $X = (N \leftthreetimes K)/K$ is commutative. We say that such $X$ is a space of a *wooden* type. Our goal is to classify commutative indecomposable homogeneous spaces of a non-wooden type.

Let $X$ be an indecomposable maximal principal commutative homogeneous space of Heisenberg type. Suppose it is not $Sp_1$-saturated, i.e., the third condition of Definition 8 is not fulfilled. Then there is a direct factor $L_i \subset K$ and an irreducible $K$-invariant subspace $\mathfrak{w}_j \subset \mathfrak{n}/\mathfrak{n}'$ such that the action $L_i : \mathfrak{w}_j$ is non-trivial, the action $(Z(L) \times L^i) : \mathfrak{w}_j$ is irreducible, and $L_j$ acts non-trivially on $(\mathfrak{n}/\mathfrak{n}')/\mathfrak{w}_i$. Clearly, $L_i = Sp_1$ and $\mathfrak{w}_i = \mathbb{H}^n$. We enlarge $K$ replacing $L_i$ by $Sp_1{\times}Sp_1$, where first $Sp_1$ acts non-trivially only on $\mathfrak{w}_i$ and the second on $(\mathfrak{n}/\mathfrak{n}')/\mathfrak{w}_i$; and replace $\mathfrak{n}$ by $\mathfrak{n}_i \oplus (\mathfrak{v}^i + [\mathfrak{v}^i, \mathfrak{v}^i])$. Note that if the intersection $\mathfrak{n}_i' \cap [\mathfrak{v}^i, \mathfrak{v}^i]$ is non-zero, then it is isomorphic to $\mathfrak{l}_i$. Repeating this procedure we obtain an $Sp_1$-saturation $\widetilde{X}$ of $X$, which is a product $\widetilde{X} = X_1 \times \ldots \times X_r$ of several principal indecomposable $Sp_1$-saturated commutative spaces of Heisenberg type. Each $X_i$ is either a maximal commutative space or a central reduction of a maximal indecomposable commutative space. Let $X_i = (\widetilde{N}_i \rtimes \widetilde{K}_i)/\widetilde{K}_i$. Then $\widetilde{\mathfrak{n}}_i/\widetilde{\mathfrak{n}}_i' \subset \mathfrak{n}/\mathfrak{n}'$ and each $\widetilde{K}_i$ contains a normal subgroup $\mathrm{Sp}_1$.

**Lemma 4.12.** *Suppose $\widetilde{\mathfrak{n}}_i$ is not Abelian, then $\widetilde{\mathfrak{n}}_i$ can be identify with a subalgebra of $\mathfrak{n}$ and $\widetilde{K}_i$ is isomorphic to a maximal connected subgroup of $K$ acting on it locally effectively.*

*Proof.* Let $\widetilde{\mathfrak{n}}_i = W \oplus \widetilde{\mathfrak{n}}_i'$ be a $\widetilde{K}_i$-invariant decomposition. Then we can consider $W$ as a subspace of $\mathfrak{n}$ such that $W \cap \mathfrak{n}' = 0$. Assume that $(W + [W, W]) \subset \mathfrak{n}$ is not isomorphic to $\widetilde{\mathfrak{n}}_i$. It means that $(W + [W, W])$ was "enlarged". Hence, there are direct factors $L_1, L_2$ of $\widetilde{K}_i$ such that $L_1 \cong \mathrm{L}_2 \cong \mathrm{Sp}_1$, and $\widetilde{\mathfrak{n}}_i'$ contains a subspace isomorphic to $\mathfrak{l}_1 \oplus \mathfrak{l}_2$. According to Tables 3.1 and 3.2, this is possible only in two cases: $\widetilde{K}_i = \mathrm{SO}_4$, $\widetilde{\mathfrak{n}}_i = \mathbb{R}^4 \oplus \mathfrak{so}_4$ and $\widetilde{K}_i = \mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{Sp}_1$, $\widetilde{\mathfrak{n}}_i = (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus (\mathbb{H}^n \oplus \mathfrak{sp}_1)$. But in both of them the action of $\mathrm{Sp}_1 \times (\widetilde{K}_i/(L_1 \times L_2))$, where $\mathrm{Sp}_1$ is a diagonal of $L_1 \times L_2$, on $\widetilde{\mathfrak{n}}_i/\mathfrak{sp}_1$ is not commutative. Thus $\widetilde{\mathfrak{n}}_i \cong (W + [W, W])$.

Assume that $\widetilde{K}_i$ is larger then the maximal connected subgroup of $K$ acting on $\widetilde{\mathfrak{n}}_i$ locally effectively. Then $(\mathrm{Sp}_1 \times \mathrm{Sp}_1) \lhd \widetilde{K}_i$ and the action of $\mathrm{Sp}_1 \times (\widetilde{K}_i/(L_1 \times L_1))$, where the first $\mathrm{Sp}_1$ is the diagonal of $(L_1 \times L_2) = (\mathrm{Sp}_1 \times \mathrm{Sp}_1)$, on $\widetilde{\mathfrak{n}}_i$ is commutative.

Let $\widetilde{\mathfrak{n}}_i/\widetilde{\mathfrak{n}}_i' = \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_r$ be the decomposition into the sum of irreducible $\widetilde{K}_i$-subspaces. If both $L_1$, $L_2$ act on $\mathfrak{w}_j$ non-trivially and $[\mathfrak{w}_j, \mathfrak{w}_j] \neq 0$, then either $\mathfrak{w}_j = \mathbb{H}$ or $\mathfrak{w}_j = \mathbb{C}^2 \otimes \mathbb{C}^2$. But in both cases the action of $\mathrm{Sp}_1 \times (\widetilde{K}_i/(L_1 \times L_2))$ on $\mathfrak{w}_j \oplus [\mathfrak{w}_j, \mathfrak{w}_j]$ is not commutative.

According to Table 3.2, the only other possebilities for $\widetilde{K}_i : \widetilde{\mathfrak{n}}_i$ are $\mathrm{U}_2 \times \mathrm{U}_2 : (\mathbb{C}^2 \oplus \mathbb{R}) \oplus \mathbb{C}^2 \otimes \mathbb{C}^2$, $\mathrm{U}_2 \times \mathrm{U}_2 \times \mathrm{U}_2 : (\mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{R}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{R})$, item 10 of Table 3.2; and central reductions of these spaces. But in each case the action of $\mathrm{Sp}_1 \times (\widetilde{K}_i/(L_1 \times L_2))$ on $\widetilde{\mathfrak{n}}_i$ is not commutative. □

In the following we consider $\widetilde{\mathfrak{n}}_i$ as a subalgebra of $\mathfrak{n}$ and $\widetilde{K}_i$ as a subgroup of $K$.

**Lemma 4.13.** *Preserve the notation introduced above. If $\widetilde{K}_i = (\mathrm{Sp}_1)^s \times H$ and for any proper subgroup $H_1 \subset (\mathrm{Sp}_1)^s$ the action $(H_1 \times H) : \widetilde{\mathfrak{n}}_i$ is not commutative, then $X$ is of wooden type.*

*Proof.* By Lemma 4.12, $\widetilde{K}_i \subset K$ and $\widetilde{\mathfrak{n}}_i \subset \mathfrak{n}$. Let $V$ be a $K$-invarint complement of $\widetilde{\mathfrak{n}}_i$ in $\mathfrak{n}$. The subgroup $H$ acts on $V$ trivially. On the other hand, $((\mathrm{Sp}_1)^s)_{\circledast}(V) = (\mathrm{Sp}_1)^s$. Thus we can construct a forest $Fr_s$. Since $X$ is indecomposable, it is of wooden type. □

**Lemma 4.14.** *If $X_i$ is listed in Table 3.2, but not in rows 9, 10, 11 or 12, then $X$ is of wooden type.*

*Proof.* Preserve the notation of Lemma 4.13. Assume that for some proper subgroup $H_1 \subset (\mathrm{Sp}_1)^s$ the action $(H_1 \times H) : \widetilde{\mathfrak{n}}_i$ is commutative. As we have seen in the proof of Lemma 4.12, $H_1$ contains no diagonals of $\mathrm{Sp}_1 \times \mathrm{Sp}_1$, hence, $H_1 \subset \mathrm{U}_1 \times (\mathrm{Sp}_1)^{s-1}$. But then the action $\widetilde{K}_i : \widetilde{\mathfrak{n}}_i \oplus \mathbb{R}^3$, where the first $\mathrm{Sp}_1$ factor of $(\mathrm{Sp}_1)^s$ acts on $\mathbb{R}^3$, is commutative. If it is $\mathrm{Sp}_1$-saturated, then $X_i$ is not listed in Table 3.2. Note that this action can be non-$\mathrm{Sp}_1$-saturated only for spaces from rows 9, 10, 11 and 12 of Table 3.2. □

Suppose $\mathfrak{n} = \widetilde{\mathfrak{n}}_i \oplus V$, $\widetilde{\mathfrak{n}}_i' \neq 0$ and $\widetilde{K}_i = (\mathrm{Sp}_1)^s \times H$. Then $(\mathrm{Sp}_1)^s_{\circledast}(V) = H_1 \times \cdots \times H_s$, where $H_j \subset \mathrm{Sp}_1$. If $H_j$ is $\mathrm{Sp}_1$ or $\mathrm{U}_1$ we can construct either a tree $T_q$ or a graph $\Gamma_r$. Similar to the proof of Lemma 4.12, one can show that two graphs corresponding to distinct direct factors of $\widetilde{K}_i$ do not intersect. Let $V(T_q) \subset V$ and $V(\Gamma_r) \subset V$ be the corresponding subspaces. Possible non-trivial Lie algebra structures on $V(T_q)$ and $V(\Gamma_r)$ are described in Lemma 4.10 and Example 13. But if $H_j$ is trivial or finite, we cannot say anything.

Recall that $K = L = L_i \times L^i \times Z(L)$. In Table 4.1 we peresent four examples of commutative spaces $(N \rtimes K)/K$ such that the action $(Z(L) \times L^i) : \mathfrak{n}$ is commutative. This direct factor $L_i$ is put into a box.

Table 4.1.

| $\boxed{\mathrm{Sp}_1} \times \mathrm{Sp}_n \ : \ (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus HS_0^2\mathbb{H}^n$ | $\boxed{\mathrm{Sp}_1} \times \mathrm{Sp}_n \times \mathrm{Sp}_m \ : \ (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus \mathbb{H}^n \otimes \mathbb{H}^m$ |
|---|---|
| $\begin{array}{ccc} \boxed{\mathrm{Sp}_1} \times \mathrm{Sp}_n \times & \mathrm{Sp}_1 & : \ (\mathbb{H}^n \oplus \mathfrak{sp}_1) \oplus \mathbb{H}^n \\ & \downarrow & \\ & T_q & \end{array}$ | $\begin{array}{ccc} \boxed{\mathrm{Sp}_1} \times & \mathrm{Sp}_1 & : \ \mathbb{H} \oplus \mathfrak{sp}_1 \\ & \downarrow & \\ & T_q & \end{array}$ |

Note that the spaces in the second row are $\mathrm{Sp}_1$-saturated only if the trees $T_q$ are trivial.

**Theorem 4.15.** *Suppose $X$ is a principal maximal indecomposable non-$\mathrm{Sp}_1$-saturated commutative space of Heisenberg type and $\mathfrak{n}' \neq 0$. If there is a $K$-invariant non-commutative subspace $\mathfrak{w}_1 \subset (\mathfrak{n}/\mathfrak{n}')$ such that the action $K_e : \mathfrak{n}_1$ is not $(\mathrm{Sp}_1 \times \mathrm{Sp}_n) : \mathbb{H} \oplus \mathfrak{sp}_1$ and $X$ is not of wooden type, then $X$ is one of the following spaces.*

$$\begin{array}{ccc} \mathrm{U}_1 \times \mathrm{SU}_2 \times \mathrm{Sp}_n : \mathbb{C}^2 \otimes \mathbb{C}^{2n} \oplus \mathbb{R} & \mathrm{SU}_2 \times (\mathrm{S})\mathrm{U}_m : \mathbb{C}^2 \otimes \mathbb{C}^m \oplus \mathbb{R} & \mathrm{SU}_2 \times \mathrm{U}_4 : (\mathbb{C}^2 \otimes \mathbb{C}^4 \oplus \mathbb{R}) \oplus \mathbb{R}^6 \\ \downarrow & \downarrow & \downarrow \\ \Gamma_q & \Gamma_q & \Gamma_q \end{array}$$

$$\begin{array}{ccc} \mathrm{U}_1 \times \mathrm{SU}_2 \ : \ \mathbb{C}^2 \oplus \mathbb{R} & \mathrm{U}_1 \times \mathrm{SU}_2 \times \mathrm{SU}_2 : \mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{R} & \mathrm{U}_1 \times \mathrm{Sp}_m \times \mathrm{Sp}_1 : (\mathbb{H}^m \oplus \mathbb{R}) \oplus \mathbb{H}^m \\ \downarrow & \downarrow \quad \downarrow & \downarrow \\ \Gamma_q & T_q \quad \Gamma_r & \Gamma_r \end{array}$$

*Here $[V(T_q), V(T_q)] = 0$ if $V(T_q) \subset \mathfrak{n}$ corresponds to a tree $T_q$ and $(V(\Gamma_q) + [V(\Gamma_q), V(\Gamma_q)]) = \mathfrak{n}(\Gamma_q)$, where $\mathfrak{n}(\Gamma_q)$ is a Lie algebra attached to $\Gamma_q$ by the rules of Lemma 4.10.*

*Proof.* Let $\widetilde{X} = X_1 \times \ldots \times X_r$ be an $\mathrm{Sp}_1$-saturation of $X$. We may assume that $\mathfrak{n}_1 \subset \widetilde{\mathfrak{n}}_1$. Let $V$ be a $K$-invariant complement of $\widetilde{\mathfrak{n}}_1$ in $\mathfrak{n}$. Suppose that $X$ is not of wooden type. Repeat the argument we used to prove Lemma 4.14. Either the action $\widetilde{K}_1 : \widetilde{\mathfrak{n}}_1 \oplus \mathbb{R}^3$ is contained in Table 3.2; or there is a subspace $\mathfrak{w}_2 = \mathbb{H}^n \otimes \mathbb{H}$ such that $\widetilde{K}_1$ acts on it as $H \times \mathrm{Sp}_1$, where $H$ is an irreducible subgroup of $\mathrm{Sp}_n$. First assumption yields five possibilities for $\widetilde{X}_1$, which are given in the first rows of the first five diagrams. If $\widetilde{K}_1$ contains only one direct factor isomorphic to $\mathrm{Sp}_1$ and $\mathrm{U}_1 \subset (\mathrm{Sp}_1)_{\circledast}(V)$, then we construct a graph $\Gamma_q$. The fifth case $\widetilde{K}_1 = \mathrm{U}_1{\times}\mathrm{Sp}_1{\times}\mathrm{Sp}_1$ is different. Here $(\mathrm{Sp}_1{\times}\mathrm{Sp}_1)_{\circledast}(V)$ contains $\mathrm{Sp}_1{\times}\mathrm{U}_1$ and we construct a tree $T_q$ and a graph $\Gamma_r$. They do not intersect, because, otherwise the group $(\mathrm{Sp}_1{\times}\mathrm{Sp}_1)_{\circledast}(V)$ would be smaller. Since $X$ is indecomposable, $V$ equals to a vector space corresponding to $\Gamma_q$ or $T_q$ and $\Gamma_r$.

Assume now that there is $\mathfrak{w}_2 = \mathbb{H}^n{\otimes}\mathbb{H} \subset (\widetilde{\mathfrak{n}}_1/\widetilde{\mathfrak{n}}_1')$ such that $\widetilde{K}_1$ acts on it as $H{\times}\mathrm{Sp}_1$ and $H$ is an irreducible subgroup of $\mathrm{Sp}_n$. Then, according to Tables 3.1 and 3.2, $H = \mathrm{Sp}_n$, $\mathfrak{n}_1$ is either $\mathbb{H}^m{\oplus}\mathbb{R}$ or $\mathbb{H}^m{\oplus}\mathbb{H}_0$, $\widetilde{\mathfrak{n}}_1$ is either $\mathfrak{n}_1 \oplus (\mathbb{H}^m{\oplus}\mathbb{H}_0)$ or $\mathfrak{n}_1 \oplus \mathbb{H}^m{\otimes}\mathbb{H}^n$ and $\widetilde{K}_i = (\mathrm{U}_1{\times})\mathrm{Sp}_m{\times}\mathrm{Sp}_n$. Thus either $n$ or $m$ equals 1.

Suppose $m = 1$. In case $K_e = \mathrm{U}_1{\times}\mathrm{Sp}_1$, $\mathfrak{n}_1 = \mathbb{H}{\oplus}\mathbb{H}_0$, $X$ is of wooden type, because the action $(\mathrm{U}_1{\times}\mathrm{U}_1) : \mathfrak{n}_1$ is not commutative. Commutative spaces $X$ with $\mathfrak{n}_1 = \mathbb{H}{\oplus}\mathbb{R}$ are described by the fourth diagram.

If $m > 1$ and $\widetilde{\mathfrak{n}}_1 = (\mathbb{H}^m{\oplus}\mathbb{R}) \oplus (\mathbb{H}^m{\oplus}\mathbb{H}_0)$, $\widetilde{K}_i = \mathrm{U}_1{\times}\mathrm{Sp}_m{\times}\mathrm{Sp}_1$, then $X$ is of wooden type by Lemma 4.13. The last case $\widetilde{K}_i = \mathrm{U}_1{\times}\mathrm{Sp}_m{\times}\mathrm{Sp}_1$, $\widetilde{\mathfrak{n}}_1 = (\mathbb{H}^m{\oplus}\mathbb{R}) \oplus \mathbb{H}^m$ is shown on the sixth diagram. □

Consider a commutative space $X$ with $K = \mathrm{Sp}_1{\times}\mathrm{Sp}_n$, $\mathfrak{n} = \mathbb{H}^n \oplus \mathfrak{sp}_1$. There are two minimal subgroups $H \subset K$ such that the action $H : \mathfrak{n}$ is commutative, namely, $(\mathrm{Sp}_1)^n \subset \mathrm{Sp}_n$ and $\mathrm{Sp}_1 \times \mathrm{U}_1{\times}(\mathrm{Sp}_1)^{n-1}$, where $\mathrm{U}_1{\times}(\mathrm{Sp}_1)^{n-1} \subset \mathrm{Sp}_n$.

**Lemma 4.16.** *Let $X$ be a maximal indecomposable principal commutative homogeneous space such that there is a Lie subalgebra $\mathfrak{n}_1 = (\mathbb{H}^n \oplus \mathfrak{sp}_1) \subset \mathfrak{n}$ with $K_e = \mathrm{Sp}_1 \times \mathrm{Sp}_n$. If $X$ is not of wooden type and $\mathrm{Sp}_1 \times \mathrm{U}_1 \times (\mathrm{Sp}_1)^{n-1} \subset (K_e)_{\circledast}(\mathfrak{v}^1) \subset \mathrm{Sp}_1 \times \mathrm{U}_1 \times \mathrm{Sp}_{n-1}$, where $\mathrm{U}_1 \times \mathrm{Sp}_{n-1} \subset \mathrm{Sp}_n$, then $X$ is one of the following spaces.*

$$\begin{array}{ccccc}
\mathrm{Sp}_1 \times \mathrm{Sp}_n \times \mathrm{Sp}_1 : \mathfrak{n}_1 \oplus \mathbb{H}^n & \qquad & \mathrm{Sp}_1 \times \mathrm{Sp}_1 : \mathfrak{n}_1 \\
\begin{array}{ccc} \downarrow & \qquad\qquad & \downarrow \\ T_q & & \Gamma_r \end{array} & & \begin{array}{cc} \downarrow & \downarrow \\ T_q & \Gamma_r \end{array}
\end{array}$$

*Proof.* Suppose first that $n = 1$, i.e., $K_e = \mathrm{Sp}_1{\times}\mathrm{Sp}_1$. By our assumptions $(\mathrm{Sp}_1{\times}\mathrm{Sp}_1)_{\circledast}(\mathfrak{v}^1) = \mathrm{Sp}_1{\times}\mathrm{U}_1$. Thus we can construct a tree $T_q$ and a graph $\Gamma_r$ with trivial intersection.

Suppose now that $n > 1$. Take a group $H \cong \mathrm{Sp}_1$ and consider an action $H \times \mathrm{Sp}_1 \times \mathrm{Sp}_n :$ $\mathbb{H}^n \oplus \mathfrak{v}^1$, where $H$ acts non-trivially only on $\mathbb{H}^n$. Clearly, $H_{\circledast}(\mathbb{H}^n \oplus \mathfrak{v}^1) = \mathrm{U}_1$ and we can construct a graph $\Gamma_{r+2}$ where $\mathrm{Sp}_{d(0)} = H$, $\mathrm{Sp}_{d(1)} = \mathrm{Sp}_n$. This graph has trivial intersection with $T_q$ corresponding to $\mathrm{Sp}_1 \subset K_e$. In particular, $(\mathrm{Sp}_1{\times}\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1) = (\mathrm{Sp}_1)_{\circledast}(\mathfrak{v}^1) \times (\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1)$.

If the vertex 1 of $\Gamma_{r+1}$ has degree 3, then $(\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1) \subset \mathrm{Sp}_1{\times}\mathrm{Sp}_{n-2} \subset \mathrm{Sp}_1{\times}\mathrm{Sp}_1{\subset}\mathrm{Sp}_{n-2}$, But this is not allowed. Similar, if there is an edge $(1, s)$ with $1 < d(s) < \infty$, then $\mathrm{Sp}_{d(s)}$ acts on $\mathfrak{v}^1/W_{1,s}$ trivially. Thus, either $(\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1) = (\mathrm{Sp}_1)^n$ or the vertex 1 of $\Gamma_{r+2}$ has degree 1 and is connected with the vertex 2 of weight 1. On the first diagram $\Gamma_r$ is a subgraph of $\Gamma_{r+2}$ containing all vertices except 0 and 1. □

Take $r$ commutative spaces $\hat{X}_i$ containing in Table 4.1. Suppose $\hat{X}_i = (\hat{N}_i \rtimes \hat{K}_i)/\hat{K}_i$ and $\hat{K}_i = \mathrm{Sp}_1 \times H_i$, where $\mathrm{Sp}_1$ is the direct factor in the box. Take any linear representation $V$ of a compact group $(\mathrm{Sp}_1)^s \times F$. Set $K := H \times H_1 \times \ldots \times H_r \times F$, where $H$ is a subgroup of $(\mathrm{Sp}_1)^r \times (\mathrm{Sp}_1)^s$, $\mathfrak{n} := \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_r \oplus V$, where $V$ is a commutative subspace, and let $X = (N \rtimes K)/K$ be a homogeneous space of $G = N \rtimes K$.

**Theorem 4.17.** *Suppose $X$ is a principal maximal indecomposable non-$\mathrm{Sp}_1$-saturated space of Heisenberg type. Then either $X$ is listed in Theorem 4.15 or Lemma 4.16, or is obtained by the described above procedure.*

*Proof.* Let $\mathfrak{w}_1$ be a non-commutative $K$-invariant subspace of $\mathfrak{n}/\mathfrak{n}'$. Assume that $K_e : \mathfrak{n}_1$ is $(\mathrm{Sp}_1 \times \mathrm{Sp}_n) : \mathbb{H}^n \oplus \mathbb{H}_0$. If this is not the case, then $X$ is listed in Theorem 4.15.

Let $L_i = \mathrm{Sp}_1$ be a direct factor of $K$, for which the third condition of Definition 8 is not satisfied. If $\mathfrak{l}_i \subset \mathfrak{n}'$, we replace $L_i$ by $\mathrm{Sp}_1 \times \mathrm{Sp}_1$ and maybe enlarge $\mathfrak{n}$. If it is not, we do nothing. We repeat this procedure as many times as possible. Let $\hat{X} = \hat{G}/\hat{K}$ be the space obtained. We have $\mathfrak{n} \subset \hat{\mathfrak{n}} \subset \widetilde{\mathfrak{n}}$. In particular, $\mathfrak{n}_1 \subset \hat{\mathfrak{n}}$. Note that, according to the construction of $\hat{X}$, the direct factor $\mathrm{Sp}_1$ of $K_e \subset \hat{K}$ acts on $(\mathfrak{n}/\mathfrak{n}')/\mathfrak{w}_1$ trivially. Clearly, $(\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1) = (\mathrm{Sp}_n)_{\circledast}(\hat{\mathfrak{n}}/\mathfrak{w}_1)$ for $\mathrm{Sp}_n \lhd K_e$. If $(\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1) \subset \mathrm{U}_1 \times \mathrm{Sp}_{n-1}$, then $X$ is listed in Lemma 4.16. We suppose that $(\mathrm{Sp}_1)^n \subset (\mathrm{Sp}_n)_{\circledast}(\mathfrak{v}^1)$. In particular, $\mathrm{Sp}_n$ corresponds to the first vertex of a tree $T_q$ (here we use we same argument as we used to prove the previous lemma). Thus, there is an indecomposable direct factor $\hat{X}_1$ of $X$, such that $\mathfrak{n}_1 \subset \hat{\mathfrak{n}}_1$ and $\hat{X}_1$ is contained in Table 4.1. Hence, we get a decomposition $\hat{X} = \hat{X}_1 \times \ldots \times \hat{X}_r \times Y$, such that $\hat{X}_i$ are commutative spaces contained in Table 4.1 and $Y = (V \rtimes ((\mathrm{Sp}_1)^s \times F))/((\mathrm{Sp}_1)^s \times F)$ is a commutative space of Euclidean type. For each space $\hat{X}_i$ there is only one direct factor $\mathrm{Sp}_1$ of $\hat{K}_i$ such that $\mathfrak{sp}_1 \subset \hat{\mathfrak{n}}_i'$. According to our construction $K$ is of the form $H \times H_1 \times \ldots \times H_r \times F$, where $H \subset (\mathrm{Sp}_1)^r \times (\mathrm{Sp}_1)^s$.

We show that the action $K : \hat{\mathfrak{n}}$ is commutative. Indeed, for each $\mathfrak{w}_i = \mathbb{H}^n$ the group $K^i$ contains $\mathrm{Sp}_n$ and the action $\mathrm{Sp}_n : \mathbb{H}^n \oplus \mathbb{H}_0$ is commutative. In general, $\mathfrak{n}$ is a central reduction of $\hat{\mathfrak{n}}$. But since $X$ is maximal $\mathfrak{n} = \hat{\mathfrak{n}} = \mathfrak{n}_1 \oplus \ldots \oplus \mathfrak{n}_r \oplus V$. □

## 4.2 Centres

Let $Z(P)$ be the connected centre of $P$. Since the action $G : (G/K)$ is assumed to be locally effective and $K$ is connected, the intersection $Z(P) \cap K$ is trivial. Clearly, $G/K$ is commutative if and only if $(G/Z(P))/K$ is commutative. In the following we suppose that $P$ is semisimple.

Let $X = G/K$ be a non-principal maximal indecomposable commutative space. We can enlarge groups $L$ and $K$ and obtain a principal commutative space $\widetilde{X}$, such that $\widetilde{L}' = L'$, $\widetilde{K}' = K'$ and $\widetilde{N} = N$. In general $\widetilde{X}$ is decomposable $\widetilde{X} = X_1 \times \cdots \times X_r$. Each $X_i =$

$(\widetilde{N}_i \leftthreetimes \widetilde{L}_i)/\widetilde{K}_i$ is a central reduction (maybe trivial) of a maximal principal indecomposable commutative space. For each $i$ either $\widetilde{L}_i$ or $\widetilde{K}_i$ has a non-trivial connected centre.

Suppose we have such a product $\widetilde{X} = X_1 \times \cdots \times X_r$. Let $C_i$ be the connected centre of $\widetilde{L}_i$ and $Z_i$ of $\widetilde{K}_i$. We have to describe all subgroups $Z(L) \subset C_1 \times \cdots \times C_r$ and $Z(K) \subset Z_1 \times \cdots \times Z_r$ such that $Z(K) \subset Z(L) \times \widetilde{L}'$ and $X = (N \leftthreetimes (Z(L) \times \widetilde{L}'))/(Z(K) \times \widetilde{K}')$ is commutative.

Let $X_1$ be either (a central reduction of) a space corresponding to row 1 of Table 1.2b or

$$\begin{tikzcd}
\mathrm{Sp}_1 \arrow[d, dash] \arrow[drr, dash] & & \mathrm{U}_{n-1} \arrow[d, dash] \\
\mathrm{Sp}_1 \arrow[d] & \times & \mathrm{SU}_n(\mathrm{SU}_{n-1,1}) \\
T_q & &
\end{tikzcd}.$$

Then $(N \leftthreetimes L)/K$, where $L = Z(L) \times \widetilde{L}'$, $K = Z(K) \times \widetilde{K}'$, is commutative if and only if $((N/\widetilde{N}_1) \leftthreetimes (L/\widetilde{L}_1))/(K/\widetilde{K}_1)$ is commutative. This statement is also true, if $X_1 = \widetilde{L}_1/\widetilde{K}_1$ is a commutative homogeneous space of a semisimple group and $\widetilde{K}_1'$ is a spherical subgroup of $\widetilde{L}_1$. In the following we suppose that $\widetilde{X}$ does not contain direct factors of these three types.

We decompose $\widetilde{X}$ in a slightly different manner, namely $\widetilde{X} = \widetilde{X}_1 \times \cdots \times \widetilde{X}_s \times \widetilde{X}_{\mathrm{Heis}} \times \widetilde{X}_{\mathrm{red}}$, where $\widetilde{X}_{\mathrm{Heis}}$ is a commutative space of Heisenberg type and $\widetilde{X}_{\mathrm{red}}$ is a commutative homogeneous space of a redactive (semisimple) group. Each $\widetilde{X}_i$ is one of the following six commutative spaces up to central reduction.

$(\mathbb{R}^{2n} \leftthreetimes \mathrm{SO}_{2n})/\mathrm{U}_n$

$$\begin{tikzcd}
\mathrm{U}_1 \arrow[d, dash] \\
\mathrm{Sp}_1 \arrow[d] \\
T_q
\end{tikzcd}$$

$((\mathbb{R}^2 \otimes \mathbb{R}^8) \leftthreetimes (\mathrm{SO}_2 \times \mathrm{SO}_8))/(\mathrm{SO}_2 \times \mathrm{Spin}_7)$

$$\begin{tikzcd}
& \mathrm{Sp}_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{Sp}_1 \arrow[d] & \times & \mathrm{Sp}_1 \arrow[d] \\
\Gamma_r & & T_q
\end{tikzcd}$$

$((H_n \leftthreetimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$

$$\begin{tikzcd}
\mathrm{Sp}_1 \arrow[d, dash] \arrow[drr, dash] & & \mathrm{Sp}_{n-1} \arrow[d, dash] \\
\mathrm{Sp}_1 \arrow[d] & \times & \mathrm{Sp}_n(\mathrm{Sp}_{n-1,1}) \\
\Gamma_q & &
\end{tikzcd}$$

Here each $\Gamma_q$ contains either a special vertex $j$ with $H(j) = \mathrm{U}_1$ or a double edge, or a triple of vertices $i, j, t$ such that $d(j) = d(t) = \infty$ and $j, t$ are connected with $i$ by edges. Note that the connected centre $C_i$ of $\widetilde{K}_i$ is one-dimensional. According to Table 1.2b, $((\mathbb{R}^2 \otimes \mathbb{R}^8) \leftthreetimes \mathrm{SO}_8)/\mathrm{Spin}_7$ is also commutative. If $\widetilde{X}_i$ is one of the last two spaces, then $(\widetilde{N}_i \leftthreetimes \widetilde{L}_i')/\widetilde{K}_i'$ is commutative if and only if $[V(\Gamma_q), V(\Gamma_q)] = 0$ and there is no special vertices such that $H(j) = \mathrm{U}_1$.

Denote by $Z_{\circledast}$ a connected central subgroup of $\widetilde{L}_{\mathrm{Heis}} = \widetilde{K}_{\mathrm{Heis}}$ such that $(\widetilde{L}_{\mathrm{Heis}})_*(\mathfrak{n})\widetilde{L}'_{\mathrm{Heis}} = Z_{\circledast} \times \widetilde{L}'_{\mathrm{Heis}}$. Let $Z(L)$ be a subgroup of $C_1 \times \cdots \times C_s \times Z(\widetilde{L}_{\mathrm{Heis}})$ and $Z(K)$ a subgroup of $Z_1 \times \cdots \times Z_s(\widetilde{K}_{\mathrm{Heis}}) \times Z(\widetilde{K}_{\mathrm{red}})$. Assume that $Z(K)$ is contained in $L := Z(L) \times \widetilde{L}'$ and set $K := Z(K) \times \widetilde{K}'$, $X = (N \leftthreetimes L)/K$.

Suppose $Z(L) = T_1 \times T_2$, where $T_2 = (Z(K)L')/(L')$. According to Theorem 1.3, $X$ is commutative if and only if $(G/T_1)/K$ is commutative and $T_1$ acts trivially on $S(\mathfrak{n})^K =$

$S(\mathfrak{n})^{T_2\times L'}$, i.e., $T_1\subset L_*L'$. In our situation, $T_1$ can be any subgroup of $Z_{\circledast}\times\prod C_i$, where the product is taken over all $i$ such that $\widetilde{X}_i$ is (a central reduction of) $((H_n\rightthreetimes \mathrm{U}_n)\times \mathrm{SU}_n)/\mathrm{U}_n$. In the following we assume that $T_1$ is trivial, i.e., we have to describe only possible centres of $K$. We make another reduction. Suppose $(\widetilde{N}_1\rightthreetimes \widetilde{L}_1')/\widetilde{K}_1'$, is commutative. Set $\hat{L}:= L\cap(\widetilde{L}_2\times\cdots\times\widetilde{L}_s\times\widetilde{L}_{\mathrm{Heis}}\times\widetilde{L}_{\mathrm{red}})$, $\hat{K}:=\hat{L}\cap K$. Then $X$ is commutative if and only if $((N/\widetilde{N}_1)\rightthreetimes\hat{L})/\hat{K}$ is commutative.

**Theorem 4.18.** *Suppose $\widetilde{X}=\widetilde{X}_1\times\cdots\times\widetilde{X}_s\times\widetilde{X}_{\mathrm{Heis}}\times\widetilde{X}_{\mathrm{red}}$ is a commutative principal homogeneous space such that there is no spherical subgroups in $\widetilde{L}_{\mathrm{red}}$ between $\widetilde{K}'_{\mathrm{red}}$ and $\widetilde{K}_{\mathrm{red}}$; and $(\widetilde{N}_i\rightthreetimes\widetilde{L}_i')/\widetilde{K}_i'$ is never commutative. Assume that $Z(L)\subset Z(K)L'$. Then $X$ is commutative if and only if $Z(K)$ is a product $T_1\times T_2$ such that*

$$T_1\subset\left(\prod_{i=1}^{s}Z_i\right)\times Z_{\circledast}\times Z(\widetilde{K}_{\mathrm{red}}),\ T_2=Z(K)\cap Z(\widetilde{K}_{\mathrm{Heis}}),\left(\prod_{i=1}^{s}Z_i\right)\times Z(\widetilde{K}_{\mathrm{red}})\subset T_1Z_{\circledast},$$

*and the action $T_2\times\widetilde{K}'_{\mathrm{Heis}}:\widetilde{\mathfrak{n}}_{\mathrm{Heis}}$ is commutative.*

*Proof.* We apply Theorem 1.3. Note that $\widetilde{X}$ satisfies all three conditions (A), (B) and (C). The equality $L=L_*K$ holds if and only if $L'=(L_*)'K'$ and $Z(L)\subset Z(K)L_*$, see [32], [34]. Clearly, $(\widetilde{L}_*)'\subset(L')_*(\mathfrak{n})$. Thus, $L'=\widetilde{L}'=(\widetilde{L}_*)'\widetilde{K}'\subset(L')_*(\mathfrak{n})K'\subset L_*K$. By our assumptions $Z(L)\subset Z(K)L'$. More precisely, $Z(L)\subset Z(K)\widetilde{L}_{\mathrm{red}}$ since $\prod_{i=1}^{s}Z_i\subset T_1Z_{\circledast}$. Because $\widetilde{L}_{\mathrm{red}}\subset L_*$, we get $Z(L)\subset Z(K)L_*$.

Condition on the subgroup $T_1$ given here is equivalent to (B). To conclude, note that $K_*(\mathfrak{m}\oplus(\bigoplus_{i=1}^{s}\widetilde{\mathfrak{n}}_i))$ acts on $\widetilde{\mathfrak{n}}_{\mathrm{Heis}}$ as $T_2\times\widetilde{K}'_{\mathrm{Heis}}$. □

It remains to describe possible connected centres of $K$ for commutative spaces of Heisenberg type. Now we return to the fist decomposition of $\widetilde{X}=X_1\times\cdots\times X_r$. Note that, $Z_{\circledast}$ is a direct product of $((\widetilde{K}_i)_*\widetilde{K}_i')/\widetilde{K}_i'$. For each indecomposable principal commutative space $\widetilde{X}_i$ one can easily calculate this group $((\widetilde{K}_i)_*\widetilde{K}_i')/\widetilde{K}_i'$. We will not do it here. Denote by $Z_{\circledast}^i$ the product of $((\widetilde{K}_j)_*\widetilde{K}_j')/\widetilde{K}_j'$ over all $j\neq i$.

**Lemma 4.19.** *In the notation of this section, homogeneou space $X$ of Heisenberg type is commutative if and only if for each $i$ such that $\widetilde{\mathfrak{n}}_i'\neq 0$ the action of $(Z(K)\cap(Z_i\times Z_{\circledast}^i))\times\widetilde{K}_i'$ on $\widetilde{\mathfrak{n}}_i$ is commutative.*

*Proof.* This readily follow from Theorem 3.1. □

# Chapter 5

# Weakly symmetric spaces

The question whether each commutative homogeneous space is weakly symmetric or not was posed by Selberg [41]. It was answered a few years ago in a paper by Lauret [26], where he constructed the first counterexample. That example is a commutative homogeneous space of Heisenberg type. On the other hand, commutative homogeneous spaces of reductive groups are weakly symmetric, see [1]. In this chapter we find out which commutative homogeneous spaces described in previous chapters are weakly symmetric.

*Example* 14. Consider commutative homogeneous space $X = (N \rtimes L)/K$ corresponding to row 4b of Table 1.2b, i.e., $L = \mathrm{SO}_8$, $K = \mathrm{Spin}_7$ and $N$ is a simply connected commutative Lie group with $\mathfrak{n} = \mathbb{R}^2 \otimes \mathbb{R}^8$. Assume that $X$ is weakly symmetric with respect to some automorphism $\sigma \in \mathrm{Aut}(G, K)$. Then, in particular, $\sigma(L) = L$. Each automorphism of $\mathrm{SO}_8$ preserving $\mathrm{Spin}_7$ is a conjugation $a(k)$ by an element $k \in \mathrm{Spin}_7$. Thus $\sigma = \sigma' a(k)$, where $\sigma'$ acts on $L$ trivially. Clearly, $X$ is weakly symmetric with respect to $\sigma'$. Note that $-\xi \in K\xi$ for each $\xi \in \mathfrak{n}$. Thus, $\sigma'$ preserves $K$-orbits in $\mathfrak{n}$ and acts on $\mathfrak{n}$ as $\pm\mathrm{id}$. Since, the image of $K$ in $\mathrm{GL}(\mathfrak{n})$ contains $-1$, we may assume that $\sigma'$ acts on $\mathfrak{n}$ as $-\mathrm{id}$.

Consider a vector $v = \eta + \xi_1 + \xi_2 \in \mathfrak{g}/\hat{\mathfrak{k}}$ such that $\eta \in \mathfrak{l}/\mathfrak{k} \cong \mathbb{R}^7$, $\xi_1 + \xi_2 \in \mathfrak{n}$ and $\xi_1$, $\xi_2$ are linear independent vectors of $\mathbb{R}^8$. Note that the stabiliser $K_{(\xi_1+\xi_2)}$ of $\xi_1 + \xi_2$ equals $K_{\xi_1} \cap K_{\xi_2} = \mathrm{Spin}_7 \cap \mathrm{SO}_6 = \mathrm{SU}_3$. If $\sigma'(v) = -kv$ for some $k \in K$, then $k\xi_1 = \xi_1$, $k\xi_2 = \xi_2$ and $k\eta = -\eta$. In particular, $k \in K_{(\xi_1+\xi_2)}$. Recall that $\mathfrak{l}/\mathfrak{k} \cong \mathbb{R}^7$ as a $K$-module. There is a non-zero $K_{(\xi_1+\xi_2)}$-invariant vector $\eta_0 \in \mathfrak{l}/\mathfrak{k}$. Clearly, vectors $v_0 = \eta_0 + \xi_1 + \xi_2$ and $-\sigma'(v)$ does not lie in the same $K$-orbit. Thus, $X$ is not weakly symmetric.

Let $X$ be a commutative space from Example 14 and $\mu$ a $G$-invariant Riemannian metric on $X$. The metric $\mu$ is defined by an element $b \in \mathbb{B}(\mathfrak{g}/\mathfrak{k})$, i.e., by a $K$-invariant scalar product on $\mathfrak{g}/\mathfrak{k}$. Note that each $b$ is also $(\mathrm{SO}_2 \times K)$-invariant. Hence, the isometry group of $X$ contains $N \rtimes (L \times \mathrm{SO}_2)$. We will see below, that $X$ is a weakly symmetric homogeneous space of $N \rtimes (L \times \mathrm{SO}_2)$. Thus, $X$ is a weakly symmetric Riemannian manifold regardless of the choice of a $G$-invariant metric.

Let $X$ be a commutative homogeneous space of Heisenberg type with irreducible action

$K : \mathfrak{n}/\mathfrak{n}'$, i.e., a space from Table 3.1. Then $\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{n}$. The commutation operation $\mathfrak{w} \times \mathfrak{w} \to \mathfrak{z}$ is determined by the condition of $K$-equivariance up to a conjugation by elements of the centraliser $Z_{\mathrm{SO}(\mathfrak{w})}(K)$. This means, that there is only one embedding $\mathfrak{z} \hookrightarrow \bigwedge^2 \mathfrak{w}$ up to the action of $Z_{\mathrm{SO}(\mathfrak{w})}(K)$.

**Lemma 5.1.** *Suppose $X = (N \rightthreetimes L)/K$ is commutative. Then there is a Weyl involution $\theta$ of $L$ such that $\theta(K) = K$ and $\theta$ acts on $\mathfrak{n}$ as an automorphism of a Lie algebra.*

*Proof.* Let $\theta$ be a Weyl involution of $L$ preserving $K$. It exists, since $(L, K)$ is a spherical pair, see [1]. Let $\mathfrak{w}_i \subset (\mathfrak{n}/\mathfrak{n}')$ be an irreducible $L$-invariant subspace. Since $\mathfrak{w}_i$ is a self-dual representation of $L$, we can define an $L$-equivariant action of $\theta$ on $\mathfrak{w}_i$. Suppose that $[\mathfrak{w}_i, \mathfrak{w}_i] = \mathfrak{z}_i \neq 0$. We have to show that there is an $L$-invariant subspace $\mathfrak{a} \subset \bigwedge^2 \mathfrak{w}_i$ such that $\mathfrak{a} \cong \mathfrak{z}_i$ as an $L$-module and $\theta(\mathfrak{a}) = \mathfrak{a}$.

Let $\mathfrak{a} \cong \mathfrak{z}_i$ be any $L$-invariant subspace of $\bigwedge^2 \mathfrak{w}_i$. Then $\theta(\mathfrak{a}) = h \cdot \mathfrak{a}$, where $h \in Z_{\mathrm{GL}(\mathfrak{w}_i)}(L)$. Suppose $\theta$ acts on $\mathfrak{w}_i$ as a matrix $A \in \mathrm{GL}(\mathfrak{w}_i)$. If we replace $A$ by $h^{-1}A$, we get a required action of $\theta$. $\square$

We say that $\theta \in \mathrm{Aut}G$ is a Weyl involution of $G = N \rightthreetimes L$ if $\theta$ definies a Weyl involution of $G/N$. Set $\mathfrak{n}^{-\theta} := \{\xi \in \mathfrak{n} |\, \theta(\xi) = -\xi\}$.

**Lemma 5.2.** *Let $(N \rightthreetimes L)/K$ be a commutative homogebeous space. Suppose there is a Weyl involution $\theta$ of $G$ such that $L(\mathfrak{n}^{-\theta}) = \mathfrak{n}$ and for (generic) $\xi \in \mathfrak{n}^{-\theta}$ the retriction of $\theta$ to $L_\xi$ is also a Weyl involution. Then $(N \rightthreetimes L)/K$ is weakly symmetric with respect to $\theta$.*

*Proof.* We may assume that $\theta(K) = K$. If this is not the case, we replace $\theta$ by a conjugated Weyl involution $a(l)\theta a(l)^{-1}$, where $l \in L$.

The homogeneous space $X$ is weakly symmetric with respect to $\theta$ if and only if $\theta(\eta) \in K(-\eta)$ for generic $\eta \in \mathfrak{g}/\mathfrak{k}$. We have $\eta = \eta_0 + \xi$, where $\eta_0 \in \mathfrak{l}/\mathfrak{k}$ and $\xi \in \mathfrak{n}$. Since $L(\mathfrak{n}^{-\theta}) = \mathfrak{n}$, there is an element $l \in L$ such that $\theta(l\xi) = -l\xi$. According to condition (A) of Theorem 1.3, $L\xi = K\xi$. Thus, we may assume that $l \in K$ or that $\theta(\xi) = -\xi$. Then $\theta(L_\xi) = L_\xi$ and the restriction of $\theta$ is a Weyl involution of $L_\xi$ (we assume that $\xi$ is generic). Clearly, $\theta(K_\xi) = K_\xi$.

To conclude, note that $L_\xi/K_\xi$ is commutative (see condition (B) of Theorem 1.3), hence, it is weakly symmetric with respect to any Weyl involution of $L_\xi$, preserving $K_\xi$, see [1]. Thus, $\theta(\eta_0) = -\mathrm{ad}(k)\eta_0$ for some $k \in K_\xi$ and $\theta(\eta) = \theta(\eta_0) - \xi = -(\mathrm{ad}(k)\eta_0 + k\xi) = -k\eta$. $\square$

**Lemma 5.3.** *Suppose $L : V$ is either $\mathrm{SU}_n : \mathbb{C}^n$, $\mathrm{SO}_n : \mathbb{R}^n$, or $L : \mathfrak{f}$, where $F$ is a normal subgroup of a compact group $L$. Then there a Weyl involution $\theta$ of $L$ such that $L(V^{-\theta}) = V$ and for generic $\xi \in \mathfrak{n}^{-\theta}$ the retriction of $\theta$ is a Weyl involution of $L_\xi$.*

*Proof.* For the first case, we set $\theta(A) = \overline{A}$, where $A \in \mathrm{SU}_n$ and $^{-}$ is the complex conjugation, $\theta(\eta) = \overline{\eta}$ for each $\eta \in V$. The non-zero $L$-orbits on $V$ are spheres $S^{2n-1}$, in particular, $L(V^{-\theta}) = V$ and the restriction of $\theta$ to the stabiliser $\mathrm{SU}_{n-1}$ is also the complex conjugation, i.e., a Weyl involution.

If $L = \mathrm{SO}_n$, a Weyl involution is a conjugation by a diagonal matrix $I \in \mathrm{O}_n$, which acts on $\mathbb{R}^n$ in a natural way. We assume that $I \in \mathrm{SO}_n$ for odd $n$. Here also the non-zero $L$-orbits on $V$ are spheres $S^{n-1}$ and $L(V^{-\theta}) = V$. If $\theta(\xi) = -\xi$, then the restriction of $\theta$ to $L_\xi = \mathrm{SO}_{n-1}$ is a Weyl involution.

Consider the third case. Each Weyl involution of $L$ preserves $F$. Let $\mathfrak{t} \subset \mathfrak{f}$ be a maximal torus such that $\theta|_{\mathfrak{t}} = -\mathrm{id}$. Since $L$ is compact, each $L$ (or $F$) orbit on $\mathfrak{f}$ intersect $\mathfrak{t}$. Hence, $L(\mathfrak{f}^{-\theta}) = \mathfrak{f}$. For generic $\xi \in \mathfrak{t}$, the stabiliser $L_\xi$ is locally isomorphic to $(L/F) \times T$, where $T$ is a commutative compact group. Clearly, the restriction of $\theta$ is a Weyl involution of $L_\xi$. $\square$

**Theorem 5.4.** *Each commutative space contained in Table 1.2b, but not in the row 4b, is weakly symmetric. Commutative spaces* $((H_n \leftthreetimes \mathrm{U}_n) \times \mathrm{SU}_n)/\mathrm{U}_n$ *and* $((\mathbb{R}^n \leftthreetimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$ *are also weakly symmetric.*

*Proof.* Let $X = (N \leftthreetimes L)/K$ be one of these commutative spaces. One have to check, that conditions of Lemma 5.2 are satisfied for $(L/P) : \mathfrak{n}$. If $\mathfrak{n}$ is commutative and the action $(L/P) : \mathfrak{n}$ is one of the three actions listed in Lemma 5.3, then $X$ is weakly symmetric. This is the case for items 2a, 3, 4c and 4d of Table 1.2b and for $((\mathbb{R}^n \leftthreetimes \mathrm{SO}_n) \times \mathrm{SO}_n)/\mathrm{SO}_n$.

Consider three remaining cases. For $(L/P) : \mathfrak{n} = (\mathrm{S})\mathrm{U_n} : (\mathbb{C}^{\mathrm{n}} \oplus \mathbb{R})$ we take the same $\theta$ as in Lemma 5.3, i.e., $\theta(A) = \overline{A}$, $\theta(\eta) = \overline{\eta}$ for each $\eta \in \mathbb{C}^n$. Here $\theta(\xi) = -\xi$ for $\xi \in \mathfrak{n}' \cong \mathbb{R}$. The rest of the proof do not difer from the proof of this case in Lemma 5.3.

Suppose $L = \mathrm{Spin}_7$, $\mathfrak{n} \cong \mathbb{R}^8$ is a commutative algebra. Here $L_* = G_2$. This space is weakly symmetric with respect to an involution $\sigma$ such that $\sigma(g) = g$ for $g \in L$ and $\sigma(\xi) = -\xi$ for $\xi \in \mathfrak{n}$. Indeed, take $\eta = \eta_0 + \xi \in \mathfrak{g}/\mathfrak{k}$, where $\eta_0 \in \mathfrak{l}/\mathfrak{k}$ and $\xi \in \mathfrak{n}$. Assume that $L_\xi = L_* = G_2$. The group $G_2$ has no auter automorphisms, so identity map is a righteous automorphism of $G_2/K_*$. Hence, there is an element $k \in K_*$ such that $\mathrm{ad}(k)\eta_0 = -\eta_0$ and $\mathrm{ad}(k)\sigma(\eta) = \mathrm{ad}(k)\eta_0 - \xi = -\eta$. Here $\theta$ is an inner automorphism of $L$, hence, $X$ is also weakly symmetric with respect to $\theta$.

The last case is 4a, where $L = \mathrm{SO}_8 \times \mathrm{SO}_2$ and either $\mathfrak{n} \cong \mathbb{R}^8 \otimes \mathbb{R}^2$, then it is commutative, or $\mathfrak{n} = \mathfrak{h}_8$. Take $\theta = \theta_1 \times \theta_2$, where $\theta_i$ are Weyl involutions of $\mathrm{SO}_8$ and $\mathrm{O}_2$, respectively. Then $\theta|_{\mathfrak{n}'} = -\mathrm{id}$. We have $\mathfrak{n}/\!\!/L = \mathfrak{n}/L \cong \mathbb{R}^2 \times \mathfrak{n}'$. For any $L$-orbit $L\xi \subset \mathfrak{n}$ where is a vector $\xi_0 \in L\xi$ such that $\theta(\xi_0) = -\xi_0$. If $\theta(\xi_0) = -\xi_0$, then the restriction of $\theta$ is a Weyl involution of $L_{\xi_0}$. Here $L_{\xi_0} = \mathrm{SO}_6$ if $L\xi$ is a generic orbit. $\square$

Thus, in class of $\mathrm{Sp}_1$-saturated principal maximal commutative spaces our task is reduced to spaces of Heisenberg type. Note that spaces of Euclidian type are symmetric. We suppose that $[\mathfrak{n}, \mathfrak{n}] \neq 0$.

Let $X = G/K$ be a commutative homogeneous space of Heisenberg type.

**Theorem 5.5.** *If* $\mathfrak{n}$ *is a direct sum of several* $K$*-invariant Hiesenberg algebras, then* $X$ *is weakly symmetric.*

*Proof.* Set $H := K(\mathbb{C})$. Since there is a non-degenerate skew-symmetric $K$-invariant bilinear form on $\mathfrak{n}/\mathfrak{n}'$, we have an isomorphism of $H$-modules $\mathfrak{n}(\mathbb{C}) \cong W \oplus W^* \oplus m\mathbb{C}$, where $\mathbb{C}$ is a trivial $H$-module. Moreover, the induced Lie algebra structure on $\mathfrak{n}(\mathbb{C})$ satisfies the following equalities: $[W, W] = [W^*, W^*] = [\mathfrak{n}(\mathbb{C}), m\mathbb{C}] = 0$, $[W, W^*] = m\mathbb{C}$. The action $H : \mathbb{C}[W]$ is multiplicity free, see [3], i.e., $W$ is a spherical representation of $H$.

Let $\mathbb{C}[W] = \bigoplus\limits_{\lambda\in\Gamma(W)} V_\lambda$ be the decomposition into the direct sum of irreducible $H$-invariant subspaces. Then

$$\mathbb{C}[\mathfrak{n}(\mathbb{C})]^H = \mathbb{C}[W \oplus W^*]^H \otimes \mathbb{C}[x_1, \ldots, x_m] = \bigoplus_{\lambda\in\Gamma(W)} (V_\lambda \otimes V_\lambda^*)^H \otimes \mathbb{C}[x_1, \ldots, x_m],$$

where $x_i$ are linear functions on the centre of $\mathfrak{n}(\mathbb{C})$.

Define an action of $\hat{H} := \mathbb{C}^* \times H$ on $W \oplus W^*$ by $z(v_1 + v_2) = zv_1 + z^{-1}v_2$, for each $z \in \mathbb{C}^*$, $v_1 \in W$, $v_2 \in W^*$. Clearly, this action extends to an action on the Lie algebra $\mathfrak{n}(\mathbb{C})$. Let $\hat{K} : \mathfrak{n}$ be a real form of $\hat{H} : \mathfrak{n}(\mathbb{C})$, in particular, $\hat{K} = \mathrm{U}_1 \times K$.

Let $\theta$ be a Weyl involution of $\hat{H}$ preserving $\hat{K}$. Then $\theta(K) = K$. We can define an action of $\theta$ on the Lie algebra $\mathfrak{n}$. Then $\theta$ acts also on $\mathfrak{n}(\mathbb{C})$ and $\theta(W) = W^*$. According to [37, Proposition 1], $\theta$ acts trivially on $\mathbb{C}[W \oplus W^*]^H$. In particular, $\theta$ preserves vectors in $(W \otimes W^*)^H \subset (S^2(W^* \oplus W))^H$. Recall that each $x_i$ is an $H$-invariant vector in $(W^* \otimes W) \subset \bigwedge^2(W^* \oplus W)$. Since $\theta(W) = W^*$, we have $\theta(x_i) = -x_i$. Clearly, $H$-invariants in $V_\lambda \otimes V_\lambda^*$ are of even degree and each $x_i$ is of the odd degree 1. Thus $\theta(f) = (-1)^{\deg f} f$ for each homogeneous $H$-invariant polinomial $f \in \mathbb{C}[\mathfrak{n}(\mathbb{C})]$. We conclude that $X$ is weakly symmetric with respect to $\theta$. □

In [9] similar statement is proved for $K = \mathrm{U}_n$, $N = H_n$. It is also shown there that $X = (N \leftthreetimes K)/K$ is weakly symmetric in the following five cases: $K = \mathrm{U}_1 \times \mathrm{Sp}_n$, $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{C}$; $K = \mathrm{Sp}_n \times \mathrm{Sp}_1 \times \mathrm{Sp}_m$, $\mathfrak{n} = \mathbb{H}^n \oplus \mathbb{H}^m \oplus \mathbb{H}_0$, where $[\mathbb{H}^n, \mathbb{H}^n] = [\mathbb{H}^m, \mathbb{H}^m] = \mathbb{H}_0$; $K = \mathrm{Sp}_2$, $\mathfrak{n} = \mathbb{H}^2 \oplus HS_0^2\mathbb{H}^2$; $K = \mathrm{SU}_4$, $\mathfrak{n} = \mathbb{C}^4 \oplus \mathbb{R}^6$; and $K = \mathrm{Spin}_7$, $\mathfrak{n} = \mathbb{R}^8 \oplus \mathbb{R}^7$.

Suppose $\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{z}$, where $\mathfrak{w} \cong (\mathfrak{n}/\mathfrak{n}')$ and $\mathfrak{z} = \mathfrak{n}'$. Take a linear function $\alpha \in \mathfrak{z}^*$. Denote by $\hat{\alpha}$ a skew-symmetric form on $\mathfrak{w}$ given by $\hat{\alpha}(\xi, \eta) = \alpha([\xi, \eta])$ for each $\xi, \eta \in \mathfrak{w}$. Let $\operatorname{Ker}\hat{\alpha}$ be the kernel of $\hat{\alpha}$. We identify $\mathfrak{z}$ and $\mathfrak{z}^*$. For any $\beta \in \operatorname{Ker}\hat{\alpha}$ we denote a stabiliser of $\beta$ in $K_\alpha$ by $K_{\alpha,\beta}$.

**Lemma 5.6.** *Suppose there is a Weyl involution $\theta$ of $G$ such that $K(\mathfrak{z}^{-\theta}) = \mathfrak{z}$, $K_\alpha((\operatorname{Ker}\hat{\alpha})^{-\theta}) = \operatorname{Ker}\hat{\alpha}$ for each (generic) $\alpha \in \mathfrak{z}^{-\theta}$ and for each (generic) $\beta \in \operatorname{Ker}\hat{\alpha}$ the restriction of theta to $K_{\alpha,\beta}$ is also a Weyl involution. Then $G/K$ is weakly symmetric.*

*Proof.* We prove that $-\theta(\xi + \alpha) \in K(\xi + \alpha)$ for a generic vector $(\xi + \alpha) \in \mathfrak{n}$, where $\xi \in \mathfrak{w}$, $\alpha \in \mathfrak{z}$. Let $(\ ,\ )_\mathfrak{z}$ be a $K$-invariant positive definite symmetric form on $\mathfrak{z}$. We identify $\alpha$ with the linear function $(\alpha,\ )_\mathfrak{z}$ on $\mathfrak{z}$. Since $K(\mathfrak{z}^{-\theta}) = \mathfrak{z}$, we may assume that $\theta(\alpha) = -\alpha$. Let

$\mathfrak{w} = \mathfrak{w}_\alpha \oplus \operatorname{Ker}\hat{\alpha}$ be a $K_\alpha$-invariant decomposition. Suppose $\xi = \xi_0 + \eta$, where $\xi_0 \in \mathfrak{w}_\alpha$, $\eta \in \operatorname{Ker}\hat{\alpha}$. There is an element $k \in K_\alpha$ such that $\theta(k\eta) = -k\eta$.

It remains to prove that $-\theta(\xi_0) \in K_{\alpha,\eta}\xi_0$. Note that $\mathfrak{w}_\alpha \oplus \mathbb{R}\alpha$ has a $K_\alpha$-invariant structure of a Heisenberg algebra, namely, put $[\xi_1, \xi_2] = (\alpha, [\xi_1, \xi_2])_{\mathfrak{z}}\alpha$. By our assumptions $\theta$ is a Weyl involution of $K_{\alpha,\eta}$ and $\theta(\alpha) = -\alpha$. Moreover, the action $K_{\alpha,\eta} : (\mathfrak{w}_\alpha \oplus \mathbb{R}\alpha)$ is commutative, see [3], [43, §4 of Chapter 2]. It follows from the proof of Theorem 5.5, that $\xi_0$ and $-\xi_0$ lie in the same $K_{\alpha,\eta}$-orbit.

Consider the complexifications $H_{\alpha,\eta} = K_{\alpha,\eta}(\mathbb{C})$ and $\mathfrak{w}_\alpha(\mathbb{C}) = W_\alpha \oplus W_\alpha^*$, where $[W_\alpha, W_\alpha^*] = \mathbb{C}\alpha$. There is a Weyl involution $\theta_1$ of $H_{\alpha,\eta}$ preserving $K_{\alpha,\eta}$ and acting on $\mathfrak{w}_\alpha(\mathbb{C})$ such that $\theta_1(\alpha) = -\alpha$ and $\theta_1(W_\alpha) = W_\alpha^*$. Clearly, $\theta{\cdot}\theta_1$ preserves the non-degenerate skew-symmetric form $\hat{\alpha}$ on $\mathfrak{w}_\alpha(\mathbb{C})$. Hence, $(\theta{\cdot}\theta_1)(W_\alpha) = W_\alpha$ and $\theta(W_\alpha) = W_\alpha^*$. Then, according to [37, Prop. 1], $\theta$ preserves $K_{\alpha,\eta}$-orbits on $\mathfrak{w}_\alpha$ and $\theta(\xi_0) \in K_{\alpha,\eta}(-\xi_0)$. □

**Corollary.** If $\mathfrak{n}$ is a direct sum of several $K$-invariant Heisenberg algebras, then $G/K$ is weakly symmetric with respect to any Weyl involution of $G$ acting on $\mathfrak{n}'$ as $-\mathrm{id}$.

Let us consider commutative homogeneous spaces with irreducible action $K : (\mathfrak{n}/\mathfrak{n}')$, i.e., spaces from Table 3.1. Here $\mathfrak{z} = \mathfrak{n}'$ is the centre of $\mathfrak{n}$ and $\mathfrak{n} = \mathfrak{w} \oplus \mathfrak{z}$.

**Theorem 5.7.** *Table 3.1 contains only one homogeneous space which is not weakly symmetric, namely item 9 with $K = \mathrm{Sp}_n$.*

*Remark* 3. Note that the space from row 9 was the first example of a commutative, but not weakly symmetric homogeneous space, constructed by Lauret in [26].

*Proof.* According to Theorem 5.5, we have to consider only those cases where $\dim \mathfrak{z} > 1$. We apply Lemmas 5.3 and 5.6. We always assume that $\alpha \in \mathfrak{z}^*$ is a generic point. Note that the space from the second row was shown to be weakly symmetric in [9].

In cases 1, 3, 8, 12, 17, 18, 20 and 22, we have $\operatorname{Ker}\hat{\alpha} = 0$. Thus, it is enough to check that there is an involution $\theta$ of $G$ such that $K(\mathfrak{z}^{-\theta}) = \mathfrak{z}$ and the restriction of $\theta$ is a Weyl involution of $K_\alpha$. For cases 1, 8, 17, 18, 20 and 22 it follows from Lemma 5.3.

The remaining four spaces we consider case by case.

**3.** Here $\theta$ is an inner automorphism of $K$. Suppose $\alpha \in \mathfrak{z}^*$ and $\theta(\alpha) = -\alpha$. Then $\operatorname{Ker}\hat{\alpha} \cong \mathbb{R}$ and $\theta|_{\operatorname{Ker}\hat{\alpha}} = -\mathrm{id}$. Moreover, $K(\mathbb{R}\alpha) = \mathfrak{z}^*$ and the restriction of $\theta$ to $K_\alpha = \mathrm{SU}_3$ is a Weyl involution.

**5(6).** Take $\theta(A) = \overline{A}$ and $\theta(\xi) = \overline{\xi}$ for $A \in K$, $\xi \in \mathfrak{n}$. Thus $\xi|_{\mathfrak{z}} = -\mathrm{id}$. Here $\operatorname{Ker}\hat{\alpha}$ is zero if $n$ is even and $\mathbb{C}$ if $n$ is odd. Suppose $n = 2m+1$, then $K_\alpha = (\mathrm{SU}_2)^m \cdot \mathrm{U}_1$ acts on $\operatorname{Ker}\hat{\alpha} \cong \mathbb{C}$ as $\mathrm{U}_1$ and $\theta|_{\mathbb{C}}$ is a complex conjugation. Clearly, $\mathrm{U}_1(i\mathbb{R}) = \mathbb{C}$ and $K_{\alpha,\beta} = (\mathrm{SU}_2)^m$. If $n = 2m$, then $K_\alpha = (\mathrm{SU}_2)^m$ and $\operatorname{Ker}\hat{\alpha} = 0$. In both cases the restriction of $\theta$ is a Weyl involution of $K_{\alpha,\beta}$.

**9.** Here we suppose that $K = \mathrm{U}_1 \times \mathrm{Sp}_n$. Let $\theta_1 \times \theta_2 = a(g_1) \times a(g_2)$ be a Weyl involution of $\mathrm{Sp}_n \times \mathrm{Sp}_1$ such that $g_1 \in \mathrm{Sp}_n$, $g_2 \in \mathrm{Sp}_1$ and $g_2\mathrm{U}_1 g_2^{-1} = \mathrm{U}_1$. Let $\theta$ be the restriction of $\theta_1 \times \theta_2$ to $\mathrm{Sp}_n \times \mathrm{U}_1$. Recall that $\mathbb{H}_0 \cong \mathfrak{sp}_1$, in particular, $\mathrm{Sp}_n$ acts on $\mathbb{H}_0$ trivially. We have $\mathbb{H}_0 = \mathbb{R} \oplus \mathbb{C}$ as an $\mathrm{U}_1$-module. Clearly, $\theta$ preserves this decomposition and $\theta|_{\mathbb{R}} = -\mathrm{id}$, $\mathbb{H}^{-\theta} \cong \mathbb{R} \oplus \mathbb{R}$.

We have an isomorphism $HS_0^2\mathbb{H}^n \cong (\mathfrak{su}_{2n}/\mathfrak{sp}_n)$. There is a so called Cartan subspace $\mathfrak{c} \subset HS_0^2\mathbb{H}^n$ such that $\mathrm{Sp}_n\mathfrak{c} = HS_0^2\mathbb{H}^n$ and $(\mathrm{Sp}_n)_{\mathfrak{c}} = (\mathrm{Sp}_1)^n$, i.e., there is a subgroup $(\mathrm{Sp}_1)^n$ acting trivially on $\mathfrak{c}$. We may assume that $g_1 \in (\mathrm{Sp}_1)^n$. Thus $a(g_1)$ acts on $HS_0^2\mathbb{H}^n$ trivially. But $a(g_2)$ acts on it as $-\mathrm{id}$. Summing up, we get $K(\mathfrak{z}^{-\theta}) = \mathfrak{z}$. Here $Ker\hat{\alpha} = 0$ and $K_\alpha = (\mathrm{Sp}_1)^n \subset \mathrm{Sp}_n$. Since we assumed that $g_1 \subset (\mathrm{Sp}_1)^n$, the restriction of $\theta$ is a Weyl involution of $(\mathrm{Sp}_1)^n$. Note that, commutative space $(N \rtimes \mathrm{Sp}_n)/\mathrm{Sp}_n$ with $\mathfrak{n} = \mathbb{H}^n \oplus HS_0^2\mathbb{H}^n \oplus \mathbb{R}^2$ is also weakly symmetric.

**12.** In this case $\mathrm{Ker}\,\hat{\alpha} = 0$. Take $\theta = \theta_1 \times \theta_2$, where $\theta_1$ is the inversion on $\mathrm{U}_1$ and $\theta_2$ is a Weyl involution of $\mathrm{Spin}_7$, which is well known to be inner. One can easily check that $\theta_1$ acts on $\mathbb{R}$ as $-\mathrm{id}$ and on $\mathbb{R}^7$ trivially. We have $(\mathbb{R}^7)^{-\theta_2} = \mathbb{R}^3$ and $K\mathfrak{z}^{-\theta} = \mathfrak{z}$. If $\theta(\xi) = -\xi$, then $K_\xi = \mathrm{U}_1 \cdot \mathrm{Spin}_6$ and the restriction of $\theta$ to $K_\xi$ is also a Weyl involution. □

Suppose $K = \mathrm{Sp}_n$ and $\mathbb{H}^n \oplus \mathbb{H}_0 \subset \mathfrak{n}$. Then $X = (N \rtimes K)/K$ is not weakly symmetric, see [26]. Indeed, let $\sigma$ be any automorphism of $G$ preserving $N$. Then $\sigma$ acts on $\mathfrak{n}$ as an element of $\mathrm{Sp}_n \times \mathrm{Sp}_1$, in particular, $\sigma|_{\mathbb{H}_0} \neq -\mathrm{id}$. Hence, $X$ is not weakly symmetric with respect to $\sigma$. There are positive definite symmetric forms $b \in \mathbb{B}(\mathbb{H}_0)$, which are not $\mathrm{U}_1$-invariant for any $\mathrm{U}_1 \subset \mathrm{Sp}_1$. For example $3x_1^2 + 2x_2^2 + x_3^2$, where $x_1, x_2, x_3$ is an orthonormal basis of $\mathbb{H}_0 = \mathbb{R}^3$ with respect to the $\mathrm{Sp}_1$-invariant scalar product. Thus, we can choose a $G$-invariant Riemannian metric on $X$ such that $X$ is not a weakly symmetric Riemannian manifold.

On the other hand, there is always an extension $\widetilde{G}$ of $G$ such that $X = \widetilde{G}/\widetilde{K}$ is a weakly symmetric homogeneous space, namely $\widetilde{G} = N \rtimes (\mathrm{U}_1 \times \mathrm{Sp}_n)$, $\widetilde{K} = \mathrm{U}_1 \times K$.

*Example* 15. Commutative homogeneous space $X = (N \rtimes L)/K$ given in row 13 of Table 3.2 is not weakly symmetric. We suppose that $K = \mathrm{Spin}_7 \times \mathrm{SO}_2$. Assume that $X$ is weakly symmetric with respect to some $\sigma \in \mathrm{Aut}(G, K)$. Similar to Example 14, we may assume that $\sigma = a(g)$ for some element $g \in \mathrm{O}_2$, $\sigma|_{\mathbb{R}^8} = -\mathrm{id}$ and, hence, $\sigma|_{\mathfrak{n}'} = \mathrm{id}$.

Let $v + \xi + \eta$ be a vector of $\mathfrak{n}$, where $v \in \mathbb{R}^8$, $\xi \in \mathfrak{n}'$, $\eta \in \mathbb{R}^7 \otimes \mathbb{R}^2$. Then $\sigma(v) - v$, $K_v = G_2 \times \mathrm{SO}_2$. If $X$ is weakly symmetric with respect to $\sigma$, then $\sigma$ acts as $-\mathrm{id}$ on $K_v$-invariants in $\mathbb{R}[(\mathbb{R}^7 \oplus \mathbb{R}^7 \otimes \mathbb{R}^2)]$ of odd degree. We have

$$S^2\mathbb{R}^7 \otimes S^2\mathbb{R}^2 \subset \mathbb{R}^7 \otimes (\mathbb{R}^7 \otimes S^2\mathbb{R}^2) \subset \mathbb{R}^7 \otimes (\Lambda^2\mathbb{R}^7 \otimes S^2\mathbb{R}^2) \subset \mathbb{R}^7 \otimes S^2(\mathbb{R}^7 \otimes \mathbb{R}^2) \subset S^3(\mathbb{R}^7 \oplus \mathbb{R}^7 \otimes \mathbb{R}^2)$$

and $S^2\mathbb{R}^7 \otimes S^2\mathbb{R}^2$ contains a non-zer $K_v$-invariant, which is also $\mathrm{O}_2$-invariant.

Here $G$ is an isometry group of $X$ for any $G$-invariant Riemannian metric. Thus, there is no group $\hat{G}$ acting on $X$ such that $G \subset \hat{G}$ and $X$ is a weakly symmetric Riemannian homogeneous space of $\hat{G}$.

**Theorem 5.8.** *All commutative spaces contained in Table 3.2, except items 11 with $K = \mathrm{Sp}_n \times \mathrm{Sp}_m$, 12 with $K = \mathrm{Sp}_n$, 13, and 25 with $K = (\mathrm{U}_1 \times)\mathrm{SU}_4$ are weakly symmetric.*

*Proof.* In cases 3, 7, 8, 16, 19 and 20 $\mathfrak{n}$ is a direct sum of several Heisenberg algebras, hence, this homogeneous spaces are weakly symmetric by Theorem 5.5.

Suppose that $\mathfrak{n} = \mathfrak{h} \oplus V$, where $\mathfrak{h}$ is a direct sum of several $K$-invariant Heisenberg algebras and $[V, V] = 0$. If there is a Weyl involution of $K$ such that $K(V^{-\theta}) = V$ and for each (generic) $v \in V^{-\theta}$ the restriction of $\theta$ is a Weyl involution of $K_v$, then $X$ is weakly symmetric. Indeed, we may assume that $\theta$ acts as $-\mathrm{id}$ on the centre of $\mathfrak{h}$. Each (generic) vector of $\xi \in \mathfrak{n}$ is of the form $\xi_0 + v$, where $\xi_0 \in \mathfrak{h}$ and $v \in V$. We may assume that $\theta(v) = -v$, then by the corollary of Lemma 5.6, $\theta(\xi_0) \in K_v(-\xi_0)$.

In view of Lemma 5.3, this argument works in cases 1, 2, 5, 6, 15, 17, 18, 21–24; in case 14 one have to make additional calculation for $\mathrm{Spin}_7 : \mathbb{R}^8$.

Consider case 4. Recall that $\mathbb{R}^6$ is a real form of $\bigwedge^2 \mathbb{C}^4$. We have proved that the homogeneous space corresponding to item 5 of Table 3.1, i.e., to $\mathrm{SU}_4 : (\mathbb{C}^4 \oplus \bigwedge^2 \mathbb{C}^4 \oplus \mathbb{R})$, is weakly symmetric. It follows that item 4 of Table 3.2 is also weakly symmetric.

In cases 9 and 10 we apply Lemma 5.6. Here $\operatorname{Ker} \hat{\alpha} = 0$ and evidently $K(\mathfrak{z}^{-\theta}) = \mathfrak{z}$.

Homogeneous space corresponding to row 12 is commutative if and only if $K = \mathrm{Sp}_n \times \mathrm{U}_1$ or $K = \mathrm{Sp}_n \times \mathrm{Sp}_1$. Argument here does not differ from one given for $\mathrm{U}_1 \times \mathrm{Sp}_n : (\mathbb{H}^n \oplus HS_0^2\mathbb{H}^n \oplus \mathbb{H}_0)$. As was shown in Example 15, item 13 is not weakly symmetric. Consider the remaining two cases.

**11.** Suppose $K = \mathrm{Sp}_n \times (\mathrm{Sp}_1, \mathrm{U}_1) \times \mathrm{Sp}_m$. Let $\theta = a(g_1) \times a(g_2) \times a(g_3)$, where $g_1 \in \mathrm{Sp}_n$, $g_2 \in \mathrm{Sp}_1$, $g_3 \in \mathrm{Sp}_m$ be a Weyl involution of $K$. We assume that $g_2$ normalise $\mathrm{U}_1$. Set $\theta(v) = -g_1 v g_3^{-1}$ for $v \in \mathbb{H}^n \otimes \mathbb{H}^m$. One can calculate, that $(\mathbb{H}^n \otimes \mathbb{H}^m)^{-\theta} = \mathbb{H}^d$, where $d := \min(n, m)$, and $K(\mathbb{H}^n \otimes \mathbb{H}^m)^{-\theta} = (\mathbb{H}^n \otimes \mathbb{H}^m)$. We have $(\mathrm{Sp}_1)^n \times (\mathrm{U}_1, \mathrm{Sp}_1) \subset K_v$ for generic $v \in (\mathbb{H}^n \otimes \mathbb{H}^m)^{-\theta}$. We only need to know, that $K_v((\mathfrak{n}')^{-\theta}) = \mathfrak{n}'$. But this is already true for the action of $\mathrm{U}_1$ or $\mathrm{Sp}_1$. If $K = \mathrm{Sp}_n \times \mathrm{Sp}_m$, then the corresponding homogeneos space is not weakly symmetric. It follows form the fact that item 9 of Table 3.1 is not weakly symmetric for $K = \mathrm{Sp}_n$.

**25.** If $K = (\mathrm{U}_1 \times)\mathrm{SU}_4 \times \mathrm{SO}_2$, we take $\theta = \theta_1 \times \theta_2$, where $\theta_1$ is a Weyl involution of $\mathrm{SU}_4$ and $\theta_2$ of $\mathrm{SO}_2$. Then each $K$-orbit in $\mathbb{R}^6 \otimes \mathbb{R}^2$ contains a vector $\eta$ such that $\theta(\eta) = -\eta$ and the restriction of $\theta$ is a Weyl involution of $K_\eta = (\mathrm{U}_1 \times)\mathrm{Sp}_1 \times \mathrm{Sp}_1$. Clearly, $\theta|_{\mathfrak{n}'} = -\mathrm{id}$ and $X$ is weakly symmetric. If $K = (\mathrm{U}_1 \times)\mathrm{SU}_4$, then $X$ is not weakly symmetric. It can be shown in the same way as in Example 15. □

We do not consider non-principal or non-$\mathrm{Sp}_1$-saturated commutative space. For each particular $X$ one can verify whether $X$ is weakly symmetric or not following the strategy of this chapter.

# Bibliography


[1] D.N. Akhiezer, E.B. Vinberg, Weakly symmetric spaces and spherical varieties, *Transformation Groups* **4** (1999), 3–24.

[2] D.N. Akhiezer, D.I. Panyushev, Multiplicities in the branching rules and the complexity of homogeneous spaces, *Moscow Math. J.* **2** (2002), 17–33.

[3] C. Benson, J. Jenkins, and G. Ratcliff, On Gelfand pairs associated with solvable Lie groups, *Trans. AMS* **321** (1990), 85–116.

[4] C. Benson and G. Ratcliff, A classification of multiplicity free actions, *Journal of Algebra*, **181** (1996), no. 1, 152–186.

[5] F.A. Berezin, I.M. Gelfand, M.I. Graev, M.A. Naymark, Representations of groups, *Uspekhi Mat. Nauk* **11**, no. 6 (1956), 13–40 (in Russian).

[6] J. Berndt, F. Prüfer, and L. Vanhecke, Symmetric-like Riemannian manifolds and geodesic symmetries, *Proc. Royal Soc. Edinburgh, Sect. A* **125** (1995), 265–282.

[7] J. Berndt, O. Kowalski and L. Vanhecke, Geodesics in weakly symmetric spaces, *Ann. Glob. Anal. Geom.* **15** (1997), 153–156.

[8] J. Berndt, L. Vanhecke, Geometry of weakly symmetric spaces, *J. Math. Soc. Japan.* **48** (1996), 745–760.

[9] J. Berndt, F. Ricci and L. Vanhecke, Weakly symmetric groups of Heisenberg type, *Diff. Geom. Appl.* **8** (1998), no. 3, 275–284.

[10] M. Brion, Classification des espaces homogènes sphériques, *Compositio Math.* **63** (1987), 189–208.

[11] M. Brion, Représentations exceptionnelles des groupes semi-simples, *Ann. Sci. École Norm. Sup. (4)*, **18** (1985), no. 2, 345–387.

[12] É. Cartan, Geometry of Riemannian spaces. Math Sci Press, Brookline, MA, 1983.

[13] M.L. Chumak, Integrable $G$-invariant Hamiltonian systems and homogeneous spaces with a simple spectrum, *Funkts. Anal. i Prilozh.* **20** (1986), no. 4, 91–92; english translation in *Funct. Anal. Appl.* **20** (1986), 334–336.

[14] A.G. Élashvili, Canonical form and stationary subalgebras of points in general position for simple linear Lie groups, *Funkts. Anal. i Prilozh.* **6** (1972), no. 1, 51–62; english translation in *Funct. Anal. Appl.* **6** (1972), 44–53.

[15] A.G. Élashvili, Stationary subalgebras of points of general position for irreducible linear Lie groups, *Funkts. Anal. i Prilozh.* **6** (1972), no. 2, 65–78; english translation in *Funct. Anal. Appl.* **6** (1972), 139–148.

[16] J. Faraut, Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques, in: "Analyse Harmonique", Nancy, CIMPA, 1980, 315–446.

[17] G. Fels, A note on homogeneous locally symmetric spaces, *Transformation Groups* **2** (1997), no. 3, 269–277.

[18] I.M. Gelfand, Spherical functions in symmetric Riemann spaces, *Dokl. AN SSSR* **70** (1950), 5–8. English translation in I.M. Gelfand, Collected papers. Vol. II. 31–35, Gindikin et al. (Eds.) Springer-Verlag, Berlin, 1988.

[19] S. Helgason, Groups and Geometric Analysis. Acad. Press, London, 1984.

[20] S. Helgason, Differential geometry, Lie groups, and symmetric spaces. Pure and Applied Mathematics, 80. Academic Press, Inc., New York-London, 1978.

[21] S. Helgason, Differential geometry and symmetric spaces. Pure and Applied Mathematics, 12. Academic Press, Inc., New-York, 1962.

[22] Kac, V. G., Some remarks on nilpotent orbits, *J. Algebra*, **64** (1980), no. 1, 190–213.

[23] F.I. Karpelevič, Surfaces of transitivity of a semisimple subgroup of the group of motions of a symmetric space, *Dokl. AN SSSR* **93** (1953), no. 3, 401–404. (in Russian).

[24] F. Knop, Some remarks on multiplicity free spaces, Representation theories and algebraic geometry (Montreal, PQ, 1997), 301–317, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.

[25] M. Krämer, Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen, *Compositio Math.* **38** (1979), 129–153.

[26] J. Lauret, Commutative spaces which are not weakly symmetric, *Bull. London Math. Society* **30** (1998), 29–37.

[27] J. Lauret, Gelfand pairs attached to representations of compact Lie groups, *Transform. Groups* **5** (2000), no. 4, 307–324.

[28] A.S Leahy, A classification of multiplicity free representations, *J. Lie Theory* **8** (1998), 367–391.

[29] D. Luna, Slices étales, *Bull. Soc. Math. France, Paris, Mémoire* **33** (1973), 81–105.

[30] I.V. Mikityuk (Mykytyuk), On the integrability of invariant Hamiltonian systems with homogeneous configuration spaces, *Matem. Sbornik* **129** (1986), no. 4, 514–534; english translation in *Math. USSR Sbornik* **57** (1987), no. 2, 527–546.

[31] N. Nishihara, A geometric criterion for Gelfand pairs associated with nilpotent Lie groups, *J. Funct. Anal.*, **183** (2001), no.1, 148–163.

[32] A.L. Onishchik, Inclusion relations between transitive transformation groups, *Trudy MMO* **11** (1962), 199–242 (in Russian).

[33] A.L. Onishchik, Decompositions of reductive Lie groups, *Matem. Sbornik* **80** (1969), 553–599; english translation in *Math. USSR Sbornik* **9** (1969), 515–554.

[34] A.L. Onishchik, Topology of transitive transformation groups. Johann Ambrosius Barth Verlag GmbH, Leipzig, 1994.

[35] A.L. Onishchik, E.B. Vinberg, Lie groups and algebraic groups. Springer-Verlag, Berlin, 1990.

[36] D.I. Panyushev, *Complexity and rank of homogeneous spaces*, Geometriae Dedicata, **34** (1990), 249–269.

[37] D.I. Panyushev, Reductive group actions on affine varieties and their doubling, *Ann. Inst. Fourier (Grenoble)*, **45** (1995), no. 4, 929–950.

[38] D.I. Panyushev, Some amazing properties of spherical nilpotent orbits, *Math. Zeitschrift* **245** (2003), 557–580.

[39] A.M. Popov, Finite isotropy subgroups in general position of irreducible semisimple linear Lie groups, *Trudy Moskov. Mat. Obshch.*, **50** (1987), 209–248, 262; english translation in *Trans. Moscow Math. Soc.*, (1988), 205–249.

[40] L.G. Rybnikov, On commutativity of weakly commutative Riemannian homogeneous spaces, *Funkts. Anal. i Prilozh.* **37**, (2003), no. 2, 41–51; english translation in *Funct. Anal. Appl.* **37**, (2003), 114–122.

[41] A. SELBERG, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, *J. Indian Math. Soc.* **20** (1956), 47–87.

[42] E.G.F. THOMAS, An infinitesimal characterization of Gelfand pairs, *Contemporary Math.* **26** (1984), 379–385.

[43] E.B. VINBERG, Commutative homogeneous spaces and coisotropic actions, *UMN* **56** (2001), no. 1, 3–62; english translation in *Russian Math. Surveys* **56**, no.1 (2001), 1–60.

[44] E.B. VINBERG, Commutative homogeneous spaces of Heisenberg type, *Trudy MMO* **64** (2003), 54–89; english translation in *Trans. Moscow Math. Soc.* (2003), 47–80.

[45] E.B. VINBERG, B.N KIMEL'FEL'D, Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups, *Funkts. Anal. i Prilozh.* **12** (1978), no. 3, 12–19, 96; english translation in *Funct. Anal. Appl.* **12** (1978), 168–174.

[46] O.S. YAKIMOVA, Weakly symmetric spaces of semisimple Lie groups, *Vestnik MSU, Ser. 1, Matem., mekh.* (2002), no. 2, 57–60; english translation in *Moscow Univ. Math. Bull.* **57** (2002), no. 2, 37-40.

[47] O.S. YAKIMOVA, Weakly symmetric Riemannian manifolds with reductive isometry group, *Matem. Sbornik* **195** (2004), no. 4, 143–160; english translation in *Math. USSR Sbornik* **195** (2004), 599–614.

[48] O.S. YAKIMOVA, Saturated commutative spaces of Heisenberg type, *Acta Applicandae Mathematica*, **81**(2004), no. 1, 339–345.

# Abstract

Let $K \subset G$ be a compact subgroup of a real Lie group $G$. Denote by $\mathcal{D}(X)^G$ the algebra of $G$-invariant differential operators on the homogeneous space $X = G/K$. Then $X$ is called *commutative* or the pair $(G, K)$ is called a *Gelfand pair* if the algebra $\mathcal{D}(X)^G$ is commutative. Symmetric Riemannian homogeneous spaces introduced by Élie Cartan and *weakly symmetric* homogeneous spaces introduced by Selberg in [41] are commutative. In this Dissertation we prove an effective commutativity criterion and obtain the complete classification of Gelfand pairs.

If $X = G/K$ is commutative, then, up to a local isomorphism, $G$ has a factorisation $G = N \leftthreetimes L$, where $N$ is either 2-step nilpotent or abelian and $L$ is reductive with $K \subset L$, see [43]. In Chapter 1 we impose on $X$ two technical constrains: *principality* and $\mathrm{Sp}_1$-*saturation*. These conditions describe the behaviour of the connected centres $Z(L) \subset L$, $Z(K) \subset K$ and normal subgroups of $K$ and $L$ isomorphic to $\mathrm{Sp}_1$. Under these constraints, the classification problem is reduced to reductive case ($G = L$) and *Heisenberg* case ($L = K$). In Chapter 1, we describe principal commutative homogeneous spaces such that there is a simple non-commutative ideal $\mathfrak{l}_i \neq \mathfrak{su}_2$ of $\mathrm{Lie}\, L$ which is not contained in $\mathrm{Lie}\, K$.

In Chapter 2, $G$ is supposed to be reductive. In this case the notions of commutative and weakly symmetric homogeneous spaces are equivalent; moreover, weakly symmetric spaces are real forms of complex affine spherical homogeneous spaces, see Akhiezer-Vinberg [1]. Spherical affine homogeneous spaces are classified by Krämer [25] ($G$ is simple), by Brion [10] and Mikityuk [30] ($G$ is semisimple). Classifications of [10] and [30] are not complete. They describe only principal spherical homogeneous spaces. In Chapter 2, we fill in the gaps in these classifications and explicitly describe commutative homogeneous spaces of reductive groups. This chapter also contains a classification of weakly symmetric structures on $G/K$. We obtain many new examples of weakly symmetric Riemannian manifolds. Most of them are not symmetric under some particular choice of a $G$-invariant Riemannian metric.

In Chapter 3, we complete classification of principal $\mathrm{Sp}_1$-saturated commutative spaces of Heisenberg type, started by Benson-Ratcliff [3] and Vinberg [43], [44].

In Chapter 4, constraints of principality and $\mathrm{Sp}_1$-saturation are removed. Thus, all Gelfand pairs are classified.

In Chapter 5, we classify principal maximal $\mathrm{Sp}_1$-saturated weakly symmetric homogeneous spaces. The question whether each commutative homogeneous space is weakly symmetric was posed by Selberg [41]. It was answered a few years ago in a negative way by Lauret [26]. On the other hand, commutative homogeneous spaces of reductive groups are weakly symmetric, see [1]. We prove that if $X = (N \leftthreetimes K)/K$ is commutative and $N$ is a product of several Heisenberg groups, then $X$ is weakly symmetric. Several new examples of commutative, but not weakly symmetric homogeneous spaces are obtained.

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3) "On weakly commutative homogeneous spaces", UMN 57, no. 3 (2002), 171–173.
4) "On the classification of Gelfand pairs", UMN 58, no. 3 (2003), 195–196.
5) "Weakly symmetric Riemannian manifolds with reductive isometry group", Math. Sbornik 195, no. 4, 143–160.
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